THIRD ISOMORPHISM THEOREM

Prop: Let R be a ring with ideals $I, J \trianglelefteq R$. Suppose $J \subseteq I$.
Then ① I/J is ideal in R/J and

② $$(R/J)/(I/J) \cong R/I.$$

pf ①: We show I/J is ideal in R/J by showing that it is the kernel of a homomorphism. Let

$$\alpha : R/J \to R/I \text{ be defined by } \alpha(x+J) = x+I.$$

Well-definition: If $x+J = y+J$, then $x = y+z$ for some $z \in J \subseteq I$. Thus we have

$$\alpha(x+J) = x+I = (y+z)+I = y+I = \alpha(y+J).$$

Homomorphism: Let $x+J, y+J \in R/J$. Then

Ⓐ $$\begin{aligned} \alpha((x+J)+(y+J)) &= \alpha((x+y)+J) \\ &= (x+y)+I \\ &= (x+I)+(y+I) \\ &= \alpha(x+J) + \alpha(y+J) \end{aligned}$$

Ⓜ $$\begin{aligned} \alpha((x+J)(y+J)) &= \alpha(xy+J) \\ &= xy+I \\ &= (x+I)(y+I) = \alpha(x+J)\alpha(y+J). \end{aligned}$$

Kernel: We have

$$\begin{aligned}
x+J \in \ker(\alpha) &\iff \alpha(x+J) = 0_{R/I} \\
&\iff x+I = 0_R + I \\
&\iff x - 0_R \in I \\
&\iff x \in I \\
&\iff x+J \in I/J,
\end{aligned}$$

and thus $\ker(\alpha) = I/J$. $\blacksquare$

Pf ②: We prove this in a sequence of steps.

Step 1: $\varphi : R \longrightarrow (R/J)/(I/J) : x \longmapsto (x+J) + (I/J)$

is a ring homomorphism.

Indeed, we calculate

$$\begin{aligned}
\varphi(x+y) &= ((x+y)+J) + (I/J) \\
&= ((x+J)+(y+J)) + (I/J) \\
&= ((x+J) + I/J) + ((y+J) + (I/J)) \\
&= \varphi(x) + \varphi(y).
\end{aligned}$$

Step 2: φ is surjective.

For $(y+J) + (I/J) \in (R/J)/(I/J)$, we have

$$\varphi(y) = (y+J) + (I/J).$$

Step 3: $\ker(\varphi) = I$.

$$x \in \ker(\varphi) \iff \varphi(x) = 0_{(R/J)/(I/J)}$$
$$\iff (x+J) + (I/J) = 0_{R/J} + (I/J)$$
$$\iff (x+J) - 0_{R/J} \in I/J$$
$$\iff x + J \in I/J$$
$$\iff x \in I.$$

Step 4: Apply the first isomorphism theorem.

$$R/I \underset{\text{Step 3}}{=} R/\ker(\varphi) \underset{\text{1st Iso Thm}}{\cong} \operatorname{im}(\varphi) \underset{\text{step 2}}{=} (R/J)/(I/J).$$

$\blacksquare$

Exercise 1: Let $m, n \in \mathbb{Z}$.

1. When is $m\mathbb{Z} \subseteq n\mathbb{Z}$?
2. Give a simpler ring isomorphic to
$$(\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z}).$$
3. What is $m\mathbb{Z} + n\mathbb{Z}$?
4. What is $(m\mathbb{Z} + n\mathbb{Z})/m\mathbb{Z}$?