

Cycle Decompositions

Prop: Let $X \subseteq \mathbb{Z}_{>0}$ be finite. For all $\sigma \in S_X$ there are unique $[\sigma_1, \sigma_2, \dots, \sigma_k \in S_X]$ so that $\forall i [\sigma_i(x) \neq x \Rightarrow \forall_{j \neq i} [\sigma_j(x) = x]]$

① The σ_i 's are pairwise disjoint cycles, and

② For all $1 \leq i \leq k-1$ we have

$$\min \{x \in X : \sigma_i(x) \neq x\} < \min \{x \in X : \sigma_{i+1}(x) \neq x\}.$$

$\hookrightarrow \sigma_i$'s are ordered by increasing minimum nontfixed element.

Lem: If σ and τ are disjoint cycles in S_X , then $\sigma \circ \tau = \tau \circ \sigma$.

\hookrightarrow Proved already in lecture.

pf of Proposition: We proceed by induction on $|X| = n$ to prove the existence of $\sigma_1, \sigma_2, \dots, \sigma_k$.

Base Case: If $n=0$, then $X = \emptyset$.

$$\therefore \sigma \in S_X = S_\emptyset = \{\varepsilon\} \xrightarrow{\sigma \rightarrow \sigma} [\sigma = \varepsilon = \text{empty product!}]$$

Inductive Step: Suppose the statement holds for all

$Y \subseteq \mathbb{Z}_{>0}$ provided $|Y| < n$. Let $X = \left[\min \{y \in X : \sigma(y) \neq y\} \right]$ (NB: if X DNE, then $\sigma = \varepsilon = \text{empty product}$)

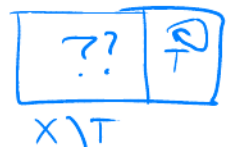
Define $T = \{x, \sigma(x), \sigma^2(x), \dots, \sigma^m(x), \sigma^{m+1}(x) = x\}$. Justify!

Observe $\sigma|_T$ is a cycle (possibly trivial) in S_T . (Hint: σ is bij. + X is finite)

$$\rightarrow \sigma|_T = (x \ \sigma(x) \ \sigma^2(x) \ \dots \ \sigma^m(x)) \quad \sigma|_T : T \rightarrow T \quad \sigma(\sigma^j(x)) = \sigma^{j+1}(x)$$

$$\text{Let } \underline{\sigma}_1(y) = \begin{cases} \sigma|_T(y) & \text{if } y \in T \\ y & \text{o/w} \end{cases}$$

This σ_1 is a cycle in S_X .



Consider $\sigma|_{X \setminus T}$. Note $\sigma|_{X \setminus T}$ is bijective.

Justify!

$\therefore \sigma|_{X \setminus T} \in \Sigma_{X \setminus T}$ and $|X \setminus T| < |X| = n$ b/c $x \in T \neq \emptyset$.

\therefore By induction $\left[\sigma|_{X \setminus T} = \bar{\sigma}_1 \bar{\sigma}_2 \dots \bar{\sigma}_{k-1} \right]$ for some $k \in \mathbb{Z}_{>0}$

\rightarrow with $\min \{ y \in X \setminus T : \bar{\sigma}_i(y) \neq y \} < \min \{ y \in X \setminus T : \bar{\sigma}_{i+1}(y) \neq y \}$

Define $\left[\bar{\sigma}_{i+1}(y) = \begin{cases} \bar{\sigma}_i(y) & \text{if } y \notin T \\ y & \text{if } y \in T \end{cases} \right]$ $\forall 1 \leq i \leq k-1$

Now ... (A) The $\bar{\sigma}_i$'s are all bijective, (B) the $\bar{\sigma}_i$'s are pairwise disjoint, and $\bar{\sigma}_i$ disjoint from $\bar{\sigma}_{i+1}$ trivially

\rightarrow (C) $\min \{ y \in X : \bar{\sigma}_i(y) \neq y \} < \min \{ y \in X : \bar{\sigma}_{i+1}(y) \neq y \}$

true of $\bar{\sigma}_2, \dots, \bar{\sigma}_k$ by the same for $\bar{\sigma}_i$, and true for $\bar{\sigma}_1$ and $\bar{\sigma}_2$ b/c $x = \min \{ y \in X : \sigma(y) \neq y \}$

Left To you: $\sigma = \bar{\sigma}_1 \bar{\sigma}_2 \dots \bar{\sigma}_k$

\therefore we have shown existence.

For uniqueness, suppose $\sigma = \alpha_1 \alpha_2 \dots \alpha_k = \beta_1 \beta_2 \dots \beta_m$ both satisfy the conditions. Note that

$$x_0 = \min \{ x : \sigma(x) \neq x \} = \min \{ x : \alpha_1(x) \neq x \} = \min \{ x : \beta_1(x) \neq x \}$$

$\therefore \left[\alpha_1^p(x_0) = \sigma^p(x_0) = \beta_1^p(x_0) \right] \forall p \in \mathbb{N}$. Justify w/ "disjoint cycles cannot"

As α_1, β_1 are cycles, we see

$$\left[\begin{aligned} \alpha_1 &= (x_0 \ \alpha_1(x_0) \ \alpha_1^2(x_0) \ \dots \ \alpha_1^j(x_0)) \\ &= (x_0 \ \beta_1(x_0) \ \beta_1^2(x_0) \ \dots \ \beta_1^j(x_0)) = \beta_1 \end{aligned} \right]$$

$$\alpha_1^{-1} \sigma = \alpha_1^{-1} \alpha_1 \alpha_2 \dots \alpha_k$$

Repeat on $\left[\alpha_2 \alpha_3 \dots \alpha_k = \alpha_1^{-1} \sigma = \beta_1^{-1} \sigma = \beta_2 \beta_3 \dots \beta_m \right]$ to obtain $\alpha_2 = \beta_2$. Repeating as necessary yields (WLOG $k \leq m$)

$$\left[\beta_{k+1} \beta_{k+2} \dots \beta_m \right] = (\alpha_1 \alpha_2 \dots \alpha_k)^{-1} \sigma = (\alpha_1 \alpha_2 \dots \alpha_k)^{-1} (\alpha_1 \alpha_2 \dots \alpha_k) = \epsilon$$

empty product! \therefore uniqueness \smile