# Cycle Decompositions

**Prop:** Let $X \subseteq \mathbb{Z}_{>0}$ be finite. For all $\sigma \in S_X$ there are unique $[\sigma_1, \sigma_2, \dots, \sigma_k \in S_X]$ so that $\forall_i [\sigma_i(x) \neq x \Rightarrow \forall_{j \neq i} [\sigma_j(x) = x]]$

① The $\sigma_i$'s are pairwise disjoint cycles, and

② For all $1 \leq i \leq k-1$ we have
$$\min\{x \in X : \sigma_i(x) \neq x\} < \min\{x \in X : \sigma_{i+1}(x) \neq x\}.$$
↳ $\sigma_i$'s are ordered by increasing minimum nonfixed element.

**Lem:** If $\sigma$ and $\tau$ are disjoint cycles in $S_X$, then $\sigma \circ \tau = \tau \circ \sigma$.

↳ Proved already in lecture.

**pf of Proposition:** We proceed by induction on $|X| = n$ to prove the existence of $\sigma_1, \sigma_2, \dots, \sigma_k$.

**Base Case:** If $n = 0$, then $X = \emptyset$.

$\therefore \sigma \in S_X = S_\emptyset = \{\varepsilon\}$ ⤳ $[\sigma = \varepsilon = \text{empty product!}]$

$\emptyset \to \emptyset$

**Inductive Step:** Suppose the statement holds for all $Y \subseteq \mathbb{Z}_{>0}$ provided $|Y| < n$. Let $x = [\min\{y \in X : \sigma(y) \neq y\}]$

(NB: if $x$ DNE, then $\sigma = \varepsilon$ = empty product).

Define $T = \{x, \sigma(x), \sigma^2(x), \dots, \sigma^m(x), \sigma^{m+1}(x) = x\}$. Justify! (Hint: $\sigma$ is bij. + $X$ is finite)

Observe $\sigma|_T$ is a cycle ~~(possibly trivial)~~ in $S_T$

$\sigma|_T : T \to T$

$\sigma(\sigma^j(x)) = \sigma^{j+1}(x)$

$$\to \sigma|_T = (x \;\; \sigma(x) \;\; \sigma^2(x) \;\; \cdots \;\; \sigma^m(x))$$

Let $\sigma_1(y) = \begin{cases} \sigma|_T(y) & \text{if } y \in T \\ y & \text{o/w} \end{cases}$. Thus $\sigma_1$ is a cycle in $S_X$.

| ?? | $\sigma\|_T$ |
|---|---|

$X \setminus T$

Consider $\sigma|_{X \setminus T}$. Note $\sigma|_{X \setminus T}$ is bijective. Justify!

$\therefore\ \sigma|_{X\setminus T} \in S_{X\setminus T}$ and $|X\setminus T| < |X| = n$ b/c $x\in T \neq \emptyset$.

$\therefore$ By induction $\left[\sigma|_{X\setminus T} = \bar\sigma_1\bar\sigma_2\cdots\bar\sigma_{k-1}\right]$ for some $k\in\mathbb{Z}_{>0}$ disjoint cycles

$\rightarrow$ with $\min\{y\in X\setminus T : \bar\sigma_i(y)\neq y\} < \min\{y\in X\setminus T : \bar\sigma_{i+1}(y)\neq y\}$

Define $\left[\sigma_{i+1}(y) = \begin{cases} \bar\sigma_i(y) & \text{if } y\notin T \\ y & \text{if } y\in T\end{cases}\right.$ glued! $\left.\forall\ 1\le i\le k-1\right]$

already defined $\sigma_1$

Now ... Ⓐ The $\sigma_i$'s are all bijective, $\leftarrow$ Justify!

Ⓑ the $\sigma_i$'s are pairwise disjoint, and $\sigma_1$ disjoint from $\sigma_i$ $(i>1)$ trivially

$\rightarrow$ Ⓒ $\min\{y\in X : \sigma_i(y)\neq y\} < \min\{y\in X : \sigma_{i+1}(y)\neq y\}$

true of $\underbrace{\sigma_2,\ldots,\sigma_k}$ by min moved elt$(\sigma_i)$ = min moved elt $(\bar\sigma_{i-1})$ the same for $\bar\sigma_i$, and

true for $\sigma_1$ and $\sigma_2$ b/c $x = \min\{y\in X : \sigma(y)\neq y\}$

$\therefore$ we have shown existence. [Left To you: $\sigma = \sigma_1\sigma_2\cdots\sigma_k$]

For uniqueness, suppose $\sigma = \alpha_1\alpha_2\cdots\alpha_k = \beta_1\beta_2\cdots\beta_m$ both satisfy the conditions. Note that

$$x_0 = \min\{x : \sigma(x)\neq x\} = \min\{x : \alpha_1(x)\neq x\} = \min\{x : \beta_1(x)\neq x\}.$$

$\therefore\ \left[\alpha_1^p(x_0) = \sigma^p(x_0) = \beta_1^p(x_0)\right]\ \forall\ p\in\mathbb{N}$. $\leftarrow$ Justify w/ "disjoint cycles commute"

As $\alpha_1, \beta_1$ are cycles, we see

$$\alpha_1 = (x_0\ \ \alpha_1(x_0)\ \ \alpha_1^2(x_0)\ \cdots\ \alpha_1^j(x_0)) = (x_0\ \ \beta_1(x_0)\ \ \beta_1^2(x_0)\ \cdots\ \beta_1^j(x_0)) = \beta_1$$

$\alpha_1^{-1}\sigma = \cancel{\alpha_1^{-1}\alpha_1}\alpha_2\cdots\alpha_k$

Repeat on $\alpha_2\alpha_3\cdots\alpha_k = \alpha_1^{-1}\sigma = \beta_1^{-1}\sigma = \beta_2\beta_3\cdots\beta_m$ to obtain $\alpha_2 = \beta_2$. Repeating as necessary yields (WLOG $k\le m$)

$$\underbrace{\beta_{k+1}\beta_{k+2}\cdots\beta_m}_{\text{empty product!}} = (\alpha_1\alpha_2\cdots\alpha_k)^{-1}\sigma = (\alpha_1\alpha_2\cdots\alpha_k)^{-1}(\alpha_1\alpha_2\cdots\alpha_k) = \varepsilon.$$

$\therefore$ uniqueness ☺

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