DERIVATIVES AS RATES OF CHANGE

150–152: The function $s(t)$ represents the position of a particle traveling along a horizontal line. Find the velocity and acceleration functions.

150. $s(t) = 2t^3 - 3t^2 - 12t + 8$

Sol: $v(t) = s'(t) = 6t^2 - 6t - 12$, $a(t) = v'(t) = 12t - 6$ ☑

151. $s(t) = 2t^3 - 15t^2 + 36t - 10$

Sol: $v(t) = s'(t) = 6t^2 - 30t + 36$, $a(t) = v'(t) = 12t - 30$ ☑

152. $s(t) = \frac{t}{1+t^2}$

Sol: $v(t) = s'(t) = \frac{(1+t^2)\cdot 1 - t\cdot(2t)}{(1+t^2)^2} = \frac{t^2+1-2t^2}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$

$$a(t) = v'(t) = \frac{(1+t^2)^2(-2t) - (1-t^2)(2(1+t^2)\cdot 2t)}{(1+t^2)^4}$$

$$= \frac{-2t(1+t^2)(1+t^2+2-2t^2)}{(1+t^2)^4} = \frac{-2t(3-t^2)}{(1+t^2)^3}$$ ☑

153. A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

Ⓐ Find the velocity of the rocket 3 seconds after being fired.

Ⓑ Find the acceleration of the rocket 3 seconds after being fired.

Sol A: $v(t) = s'(t) = -32t + 560$, so $v(3) = -32\cdot 3 + 560 = 464$ ft/s (upwards) ☑

Sol B: $a(t) = v'(t) = -32$, so $a(3) = -32$ ft/s^2 (downwards) ☑

154. A ball is thrown downward with a speed of 8 ft/s from the top of a 64 foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

(A) Determine how long it takes for the ball to hit the ground.

(B) Determine the velocity of the ball when it hits the ground.

Sol A: $s(t) = 0$ when $-16t^2 - 8t + 64 = 0$

i.e. $-8(2t^2 + t - 8) = 0$. $\therefore t = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot -8}}{2 \cdot 2} = \frac{-1 \pm \sqrt{65}}{4}$

Rejecting the negative value, the ball hits the ground after $t = \frac{-1 + \sqrt{65}}{4}$ seconds. $\boxed{\checkmark}$

Sol B: $v(t) = -32t - 8$. $\therefore v\left(\frac{-1+\sqrt{65}}{4}\right) = -32\left(\frac{-1+\sqrt{65}}{4}\right) - 8 = -8(\sqrt{65} - 2)$

So the ball hits the ground with velocity $-8(\sqrt{65} - 2)$ ft/s $\boxed{\checkmark}$

155. The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line where s is in feet and t is in seconds. In this case, $s(t) = 0$ when the back of the car is at the garage door, so $s(0) = -4$ is the starting position, i.e. 4 feet inside the garage.

(A) Determine the velocity of the car when $s(t) = 0$.

(B) Determine the velocity of the car when $s(t) = 14$.

Sol: $v(t) = s'(t) = 2t - 3$.

A: $s(t) = 0$ when $t^2 - 3t - 4 = 0$ i.e. $(t-4)(t+1) = 0$, so $t = 4$ or $\cancel{t = -1}$. $\therefore v(4) = 2 \cdot 4 - 3 = 5$ ft/s

B: $s(t) = 14$ when $t^2 - 3t - 4 = 14$, i.e. $t^2 - 3t - 18 = 0$ i.e. $(t-6)(t+3) = 0$. So $t = 6$ or $\cancel{t = -3}$

$\therefore v(6) = 2 \cdot 6 - 3 = 9$ ft/s $\boxed{\checkmark}$

156. The position of a hummingbird flying along a straight line in t seconds is given by $S(t) = 3t^3 - 7t$ meters.

(A) Determine the velocity of the bird at $t=1$ second.

(B) Determine the acceleration of the bird at $t=1$ second.

(C) Determine the acceleration of the bird when its velocity is 0 m/s.

Sol: $V(t) = S'(t) = 9t^2 - 7$, $\quad a(t) = V'(t) = 18t$.

A: $V(1) = 9 \cdot 1^2 - 7 = 2$ m/s.

B: $a(1) = 18 \cdot 1 = 18$ m/s^2.

C: $V(t) = 0$ when $9t^2 - 7 = 0$ i.e. $t^2 = \frac{7}{9}$ i.e. $t = \pm\frac{\sqrt{7}}{3}$ (Reject negative)

$\therefore a\left(\frac{\sqrt{7}}{3}\right) = 18 \cdot \frac{\sqrt{7}}{3} = 6\sqrt{7}$ m/s^2 when $V(t) = 0$.

157. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

A: Find the velocity of the potato after .5 s and 5.75 s.

Sol: $V(t) = -32t + 100$. $\quad V(.5) = 84$ m/s, $\quad V(5.75) = -84$ m/s

B: Find the speed of the potato at .5 s and 5.75 s.

Sol: $|V(.5)| = 84$ m/s, $\quad |V(5.75)| = 84$ m/s.

C: Determine when the potato reaches its maximum height.

Sol: This happens when $V(t) = 0$ (b/c the potato stops rising and starts falling).

$V(t) = 0$ when $-32t + 100 = 0$ i.e. $t = \frac{25}{8}$.

D: Find the acceleration of the potato at .5 s and 1.5 s.

Sol: $a(t) = -32$, so $a(.5) = -32$ m/s^2 and $a(1.5) = -32$ m/s^2.

$320 \times 17 \leadsto 3200 + 2240$

E: Determine how long the potato is in the air.

Sol: $S(t) = 0$ when $t = \frac{-100 \pm \sqrt{100^2 - 4 \cdot -16 \cdot 85}}{2 \cdot -16} = \frac{25 \pm \frac{1}{4}\sqrt{4^2(25^2 + 4 \cdot 85)}}{8} = \frac{25 \pm \sqrt{965}}{8}$

reject negative

$\therefore$ potato hits the ground after $\frac{25 + \sqrt{965}}{8}$ seconds

F: Determine the velocity of the potato upon hitting the ground.

Sol: $V\left(\frac{25+\sqrt{965}}{8}\right) = -32\left(\frac{25+\sqrt{965}}{8}\right) + 100 = -8\sqrt{965}$ m/s

158. The position function $S(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

A: Determine the direction the train is traveling when $S(t) = 0$.

Sol: $S(t) = 0$ when $t^3 - 8t = 0$ i.e. $t(t^2 - 8) = 0$ so $t = 0$ or $t = \pm\sqrt{8}$.

reject negative.

$V(t) = 3t^2 - 8$. $V(0) = -8$, so westward at time 0.

$V(\sqrt{8}) = 3 \cdot \sqrt{8}^2 - 8 = 16$, so eastward at time $\sqrt{8}$.

B: Determine the direction the train is traveling when $a(t) = 0$.

Sol: $a(t) = 6t$, so $a(t) = 0$ when $t = 0$. $\therefore$ Eastward by A

160. The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$ where x is the number of food processors manufactured.

$C(x) = 200 + 7x^{-1} + \frac{1}{7}x^2$

A: Calculate the marginal cost function.

Sol: $C'(x) = -7x^{-2} + \frac{2}{7}x$

B: Estimate the cost of the 13th food processor using the marginal cost function.

Sol: $C'(12) = -\frac{7}{12^2} + \frac{2}{7} \cdot 12 = \frac{24}{7} - \frac{7}{144}$ dollars.

C: Find the actual cost of manufacturing the 13th food processor.

Sol:
$$C(13) - C(12) = \left(200 + \frac{7}{13} + \frac{13^2}{7}\right) - \left(200 + \frac{7}{12} + \frac{12^2}{7}\right)$$
$$= 7\left(\frac{1}{13} - \frac{1}{12}\right) + \frac{1}{7}(169 - 144)$$
$$= \frac{-7}{156} + \frac{25}{7} \text{ dollars}$$

161. The price p in dollars and demand x for a product is given by the price-demand function $p(x) = 10 - .001x$.

A: find the revenue function

Sol: $R(x) = x \cdot p(x) = 10x - \frac{1}{1000}x^2$

B: Calculate the marginal revenue.

Sol: $R'(x) = 10 - \frac{1}{500}x$

C: Calculate marginal revenue at 2000 and 5000 units.

Sol: $R'(2000) = 10 - \frac{2000}{500} = 6$ dollars/unit.

$R'(5000) = 10 - \frac{5000}{500} = 0$ dollars/unit.

162. Profit is earned when revenue exceeds cost. The profit function for a skateboard company (RAD) is $P(x) = 30x - .3x^2 - 250$ where x is the number of skateboards sold.

A: Find the exact profit from selling the 30^{th} board.

Sol: $P(30) - P(29) = \left(30 \cdot 30 - \frac{3}{10} \cdot 30^2 - 250\right) - \left(30 \cdot 29 - \frac{3}{10} \cdot 29^2 - 250\right)$

$= 30(30-29) - \frac{3}{10}(30-29)(30+29)$

$= 30 - \frac{3}{10} \cdot 59 = 30 - 17.7 = 12.3$ dollars.

B: Use the marginal profit function to estimate the profit from selling the 30th board.

Sol: $P'(x) = 30 - \frac{3}{5}x$. $\therefore P'(29) = 30 - \frac{3}{5} \cdot 29$

$= 30 - \frac{172}{10} = 30 - 17.2 = 12.8$ dollars