

# DERIVATIVES AS RATES OF CHANGE

150-152: The function  $s(t)$  represents the position of a particle traveling along a horizontal line. Find the velocity and acceleration functions.

150.  $s(t) = 2t^3 - 3t^2 - 12t + 8$

Sol:  $v(t) = s'(t) = 6t^2 - 6t - 12$ ,  $a(t) = v'(t) = 12t - 6$   $\square$

151.  $s(t) = 2t^3 - 15t^2 + 36t - 10$

Sol:  $v(t) = s'(t) = 6t^2 - 30t + 36$ ,  $a(t) = v'(t) = 12t - 30$   $\square$

152.  $s(t) = \frac{t}{1+t^2}$

Sol:  $v(t) = s'(t) = \frac{(1+t^2) \cdot 1 - t \cdot (2t)}{(1+t^2)^2} = \frac{t^2 + 1 - 2t^2}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$

$$a(t) = v'(t) = \frac{(1+t^2)^2(-2t) - (1-t^2)(2(1+t^2) \cdot 2t)}{(1+t^2)^4}$$

$$= \frac{-2t(1+t^2)(1+t^2 + 2 - 2t^2)}{(1+t^2)^4} = \frac{-2t(3-t^2)}{(1+t^2)^3} \quad \square$$

153. A rocket is fired vertically upward from the ground. The distance  $s$  in feet that the rocket travels from the ground after  $t$  seconds is given by  $s(t) = -16t^2 + 560t$ .

(A) Find the velocity of the rocket 3 seconds after being fired.

(B) Find the acceleration of the rocket 3 seconds after being fired.

Sol/A:  $v(t) = s'(t) = -32t + 560$ , so  $v(3) = -32 \cdot 3 + 560 = 464$  ft/s (upwards)  $\square$

Sol/B:  $a(t) = v'(t) = -32$ , so  $a(3) = -32$  ft/s<sup>2</sup> (downwards)  $\square$

154. A ball is thrown downward with a speed of 8 ft/s from the top of a 64 foot-tall building. After  $t$  seconds, its height above the ground is given by  $s(t) = -16t^2 - 8t + 64$ .

(A) Determine how long it takes for the ball to hit the ground.

(B) Determine the velocity of the ball when it hits the ground.

Sol A:  $s(t) = 0$  when  $-16t^2 - 8t + 64 = 0$

i.e.  $-8(2t^2 + t - 8) = 0$ .  $\therefore t = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot -8}}{2 \cdot 2} = \frac{-1 \pm \sqrt{65}}{4}$

Rejecting the negative value, the ball hits the ground

after  $t = \frac{-1 + \sqrt{65}}{4}$  seconds.  $\square$

Sol B:  $v(t) = -32t - 8$ .  $\therefore v\left(\frac{-1 + \sqrt{65}}{4}\right) = -32\left(\frac{-1 + \sqrt{65}}{4}\right) - 8 = -8(\sqrt{65} - 2)$

So the ball hits the ground with velocity  $-8(\sqrt{65} - 2)$  ft/s  $\square$

155. The position function  $s(t) = t^2 - 3t - 4$  represents the position of the back of a car backing out of a driveway and then driving in a straight line where  $s$  is in feet and  $t$  is in seconds. In this case,  $s(t) = 0$  when the back of the car is at the garage door, so  $s(0) = -4$  is the starting position, i.e. 4 feet inside the garage.

(A) Determine the velocity of the car when  $s(t) = 0$ .

(B) Determine the velocity of the car when  $s(t) = 14$ .

Sol:  $v(t) = s'(t) = 2t - 3$ .

A:  $s(t) = 0$  when  $t^2 - 3t - 4 = 0$  i.e.  $(t - 4)(t + 1) = 0$ ,  
so  $t = 4$  or  $t = -1$ .  $\therefore v(4) = 2 \cdot 4 - 3 = 5$  ft/s

B:  $s(t) = 14$  when  $t^2 - 3t - 4 = 14$ , i.e.  $t^2 - 3t - 18 = 0$   
i.e.  $(t - 6)(t + 3) = 0$ . So  $t = 6$  or  $t = -3$

$\therefore v(6) = 2 \cdot 6 - 3 = 9$  ft/s

$\square$

156. The position of a hummingbird flying along a straight line in  $t$  seconds is given by  $s(t) = 3t^3 - 7t$  meters.

- (A) Determine the velocity of the bird at  $t=1$  second.  
 (B) Determine the acceleration of the bird at  $t=1$  second.  
 (C) Determine the acceleration of the bird when its velocity is 0 m/s.

Sol:  $v(t) = s'(t) = 9t^2 - 7$ ,  $a(t) = v'(t) = 18t$ .

A:  $v(1) = 9 \cdot 1^2 - 7 = 2$  m/s.

B:  $a(1) = 18 \cdot 1 = 18$  m/s<sup>2</sup>.

C:  $v(t) = 0$  when  $9t^2 - 7 = 0$  i.e.  $t^2 = \frac{7}{9}$  i.e.  $t = \pm \frac{\sqrt{7}}{3}$  (reject negative)  
 $\therefore a\left(\frac{\sqrt{7}}{3}\right) = 18 \cdot \frac{\sqrt{7}}{3} = 6\sqrt{7}$  m/s<sup>2</sup> when  $v(t) = 0$ .  $\square$

157. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot building. The distance in feet that the potato travels from the ground after  $t$  seconds is given by  $s(t) = -16t^2 + 100t + 85$ .

A: Find the velocity of the potato after .5 s and 5.75 s.

Sol:  $v(t) = -32t + 100$ .  $v(.5) = 84$  m/s,  $v(5.75) = -84$  m/s  $\square$

B: Find the speed of the potato at .5 s and 5.75 s.

Sol:  $|v(.5)| = 84$  m/s,  $|v(5.75)| = 84$  m/s.  $\square$

C: Determine when the potato reaches its maximum height.

Sol: This happens when  $v(t) = 0$  (b/c the potato stops rising and starts falling).  
 $v(t) = 0$  when  $-32t + 100 = 0 \Rightarrow$  i.e.  $t = \frac{25}{8}$ .  $\square$

D: Find the acceleration of the potato at .5 s and 1.5 s.

Sol:  $a(t) = -32$ ,  $\therefore a(.5) = -32$  m/s<sup>2</sup> and  $a(1.5) = -32$  m/s<sup>2</sup>.  $\square$

E: Determine how long the potato is in the air.

Sol:  $s(t) = 0$  when  $t = \frac{-100 \pm \sqrt{100^2 - 4 \cdot (-16) \cdot 85}}{2 \cdot (-16)} = \frac{25 \pm \frac{1}{4} \sqrt{4^2(25^2 + 4 \cdot 85)}}{8} = \frac{25 \pm \sqrt{965}}{8}$

$\therefore$  potato hits the ground after  $\frac{25 + \sqrt{965}}{8}$  seconds  $\square$  reject negative

F: Determine the velocity of the potato upon hitting the ground.

Sol:  $v\left(\frac{25 + \sqrt{965}}{8}\right) = -32\left(\frac{25 + \sqrt{965}}{8}\right) + 100 = -8\sqrt{965}$  m/s  $\square$

158. The position function  $S(t) = t^3 - 8t$  gives the position in miles of a freight train where east is the positive direction and  $t$  is measured in hours.

A: Determine the direction the train is traveling when  $S(t) = 0$ .

Sol:  $S(t) = 0$  when  $t^3 - 8t = 0$  i.e.  $t(t^2 - 8) = 0$  so  $t = 0$  or  $t = \pm\sqrt{8}$ .

$V(t) = 3t^2 - 8$ .  $V(0) = -8$ , so westward at time 0.

$V(\sqrt{8}) = 3 \cdot \sqrt{8}^2 - 8 = 16$ , so eastward at time  $\sqrt{8}$ .  $\square$

↑  
reject negative.

B: Determine the direction the train is traveling when  $a(t) = 0$ .

Sol:  $a(t) = 6t$ , so  $a(t) = 0$  when  $t = 0$ .  $\therefore$  Eastward by A  $\square$

160. The cost function, in dollars, of a company that manufactures food processors is given by  $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$  where  $x$  is the number of food processors manufactured.

$$C(x) = 200 + 7x^{-1} + \frac{1}{7}x^2$$

A: Calculate the marginal cost function.

Sol:  $C'(x) = -7x^{-2} + \frac{2}{7}x$   $\square$

B: Estimate the cost of the 13th food processor using the marginal cost function.

Sol:  $C'(12) = -\frac{7}{12^2} + \frac{2}{7} \cdot 12 = \frac{24}{7} - \frac{7}{144}$  dollars.  $\square$

C: Find the actual cost of manufacturing the 13th food processor.

Sol:  $C(13) - C(12) = \left(200 + \frac{7}{13} + \frac{13^2}{7}\right) - \left(200 + \frac{7}{12} + \frac{12^2}{7}\right)$   
 $= 7\left(\frac{1}{13} - \frac{1}{12}\right) + \frac{1}{7}(169 - 144)$   
 $= \frac{-7}{156} + \frac{25}{7}$  dollars  $\square$

161. The price  $p$  in dollars and demand  $x$  for a product is given by the price-demand function  $p(x) = 10 - .001x$ .

A: Find the revenue function

Sol:  $R(x) = x \cdot p(x) = 10x - \frac{1}{1000}x^2$   $\square$

B: Calculate the marginal revenue.

Sol:  $R'(x) = 10 - \frac{1}{500}x$   $\square$

C: Calculate marginal revenue at 2000 and 5000 units.

Sol:  $R'(2000) = 10 - \frac{2000}{500} = 6$  dollars/unit.

$R'(5000) = 10 - \frac{5000}{500} = 0$  dollars/unit.  $\square$

162. Profit is earned when revenue exceeds cost. The profit function for a skateboard company (RAD) is  $P(x) = 30x - .3x^2 - 250$  where  $x$  is the number of skateboards sold.

A: Find the exact profit from selling the 30<sup>th</sup> board.

Sol:  $P(30) - P(29) = (30 \cdot 30 - \frac{3}{10} \cdot 30^2 - 250) - (30 \cdot 29 - \frac{3}{10} \cdot 29^2 - 250)$   
 $= 30(30 - 29) - \frac{3}{10}(30 - 29)(30 + 29)$   
 $= 30 - \frac{3}{10} \cdot 59 = 30 - 17.7 = 12.3$  dollars.  $\square$

B: Use the marginal profit function to estimate the profit from selling the 30<sup>th</sup> board.

Sol:  $P'(x) = 30 - \frac{3}{5}x$ .  $\therefore P'(29) = 30 - \frac{3}{5} \cdot 29$   
 $= 30 - \frac{17.2}{10} = 30 - 17.2 = 12.8$  dollars  $\square$