

CHAIN RULE

214-219: Given $y = f(u)$ and $u = g(x)$, calculate $\frac{dy}{dx}$.

214. $y = 3u - 6$, $u = 2x^2$

Sol: $\frac{dy}{du} = 3$, $\frac{du}{dx} = 4x$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 4x = 12x$ \square

215. $y = 6u^3$, $u = 7x - 4$

Sol: $\frac{dy}{du} = 6 \cdot 3u^2 = 18u^2$, $\frac{du}{dx} = 7$,

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 18(7x-4)^2 \cdot 7 = 126(7x-4)^2$ \square

216. $y = \sin(u)$, $u = 5x - 1$

Sol: $\frac{dy}{du} = \cos(u)$, $\frac{du}{dx} = 5$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(5x-1) \cdot 5 = 5\cos(5x-1)$ \square

217. $y = \cos(u)$, $u = -\frac{x}{8} = -\frac{1}{8}x$

Sol: $\frac{dy}{du} = -\sin(u)$, $\frac{du}{dx} = -\frac{1}{8}$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(-\frac{1}{8}x) \cdot -\frac{1}{8} = \frac{1}{8}\sin(-\frac{1}{8}x)$ \square

218. $y = \tan(u)$, $u = 9x + 2$

Sol: $\frac{dy}{du} = \sec^2(u)$, $\frac{du}{dx} = 9$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2(9x+2) \cdot 9 = 9\sec^2(9x+2)$ \square

219. $y = \sqrt{4u+3}$, $u = x^2 - 6x$

Sol: $y = (4u+3)^{1/2}$, $\frac{dy}{du} = \frac{1}{2}(4u+3)^{-1/2} \cdot 4 = 2(4u+3)^{-1/2}$, $\frac{du}{dx} = 2x - 6$

$\frac{dy}{dx} = 2(4(x^2-6x)+3)^{-1/2} (2x-6) = \frac{4(x-3)}{\sqrt{4x^2-24x+3}}$ \square

220-227: Decompose each $y(x)$ as $y=f(u)$ for $u=g(x)$, and take the derivative via the chain rule.

220: $y = (3x-2)^6$

Sol: $\begin{cases} y = u^6 \\ y' = 6u^5 \end{cases}, \quad \begin{cases} u = 3x-2 \\ u' = 3 \end{cases} \rightarrow \frac{dy}{dx} = 6(3x-2)^5 \cdot 3 = 18(3x-2)^5 \quad \square$

221: $y = (3x^2+1)^3$

Sol: $\begin{cases} y = u^3 \\ y' = 3u^2 \end{cases}, \quad \begin{cases} u = 3x^2+1 \\ u' = 6x \end{cases} \rightarrow \frac{dy}{dx} = 3(3x^2+1)^2 \cdot 6x = 18x(3x^2+1)^2 \quad \square$

222: $y = \sin^5(x)$

Sol: $\begin{cases} y = u^5 \\ y' = 5u^4 \end{cases}, \quad \begin{cases} u = \sin(x) \\ u' = \cos(x) \end{cases} \rightarrow \frac{dy}{dx} = 5(\sin(x))^4 \cdot \cos(x) = 5\sin^4(x)\cos(x) \quad \square$

223: $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Sol: $\begin{cases} y = u^7 \\ y' = 7u^6 \end{cases}, \quad \begin{cases} u = \frac{1}{7}x + 7x^{-1} \\ u' = \frac{1}{7} - 7x^{-2} \end{cases} \rightarrow \frac{dy}{dx} = 7\left(\frac{1}{7}x + 7x^{-1}\right)^6 \left(\frac{1}{7} - 7x^{-2}\right) = (1-7x^{-2})\left(\frac{x}{7} + \frac{7}{x}\right)^6 \quad \square$

224: $y = \tan(\sec(x))$

Sol: $\begin{cases} y = \tan(u) \\ y' = \sec^2(u) \end{cases}, \quad \begin{cases} u = \sec(x) \\ u' = \sec(x)\tan(x) \end{cases} \rightarrow \frac{dy}{dx} = \sec^2(\sec(x)) \cdot \sec(x)\tan(x) \quad \square$

225: $y = \csc(\pi x+1)$

Sol: $\begin{cases} y = \csc(u) \\ y' = -\csc(u)\cot(u) \end{cases}, \quad \begin{cases} u = \pi x+1 \\ u' = \pi \end{cases} \rightarrow \frac{dy}{dx} = -\csc(\pi x+1)\cot(\pi x+1) \cdot \pi \quad \square$

226: $y = \cot^2(x)$

Sol: $\begin{cases} y = u^2 \\ y' = 2u \end{cases}, \quad \begin{cases} u = \cot(x) \\ u' = -\csc^2(x) \end{cases} \rightarrow \frac{dy}{dx} = 2\cot(x)(-\csc^2(x)) = -2\csc^2(x)\cot(x) \quad \square$

227: $y = -6(\sin(x))^{-3}$

Sol: $\begin{cases} y = -6u^{-3} \\ y' = -6 \cdot -3u^{-4} = 18u^{-4} \end{cases}, \quad \begin{cases} u = \sin(x) \\ u' = \cos(x) \end{cases} \rightarrow \frac{dy}{dx} = 18(\sin(x))^{-4} \cos(x) \quad \square$

228-237: Calculate $\frac{dy}{dx}$.

228. $y = (3x^2 + 3x - 1)^4$

Sol: $\frac{dy}{dx} = 4(3x^2 + 3x - 1)^3 (6x + 3)$
 $= 12(2x + 1)(3x^2 + 3x - 1)^3$ \square

229. $y = (5 - 2x)^{-2}$

Sol: $\frac{dy}{dx} = -2(5 - 2x)^{-3} (-2)$
 $= 4(5 - 2x)^{-3}$ \square

230. $y = \cos^3(\pi x)$

Sol: $\frac{dy}{dx} = 3\cos^2(\pi x) \cdot (-\sin(\pi x) \cdot \pi)$
 $= -3\pi \cos^2(\pi x) \sin(\pi x)$ \square

231. $y = (2x^3 - x^2 + 6x + 1)^3$

Sol: $\frac{dy}{dx} = 3(2x^3 - x^2 + 6x + 1)^2 (6x^2 - 2x + 6)$
 $= 6(3x^2 - x + 3)(2x^3 - x^2 + 6x + 1)^2$ \square

232. $y = \frac{1}{\sin^2(x)} = (\sin(x))^{-2}$

Sol: $\frac{dy}{dx} = -2(\sin(x))^{-3} \cos(x)$
 $= \frac{-2\cos(x)}{\sin^3(x)}$ \square

$$\underline{233.} \quad y = (\tan(x) + \sin(x))^{-3}$$

$$\underline{\text{Sol:}} \quad \frac{dy}{dx} = -3(\tan(x) + \sin(x))^{-4} (\sec^2(x) + \cos(x)) \quad \square$$

$$\underline{234.} \quad y = x^2 \cos^4(x)$$

$$\underline{\text{Sol:}} \quad \frac{dy}{dx} = 2x \cos^4(x) + x^2 (4 \cos^3(x) (-\sin(x))) \\ = 2x \cos^3(x) (\cos(x) - 2x \sin(x)) \quad \square$$

$$\underline{235.} \quad y = \sin(\cos(7x))$$

$$\underline{\text{Sol:}} \quad \frac{dy}{dx} = \cos(\cos(7x)) (-\sin(7x) \cdot 7) \\ = -7 \cos(\cos(7x)) \sin(7x) \quad \square$$

$$\underline{236.} \quad y = \sqrt{6 + \sec(\pi x^2)} = (6 + \sec(\pi x^2))^{1/2}$$

$$\underline{\text{Sol:}} \quad \frac{dy}{dx} = \frac{1}{2} (6 + \sec(\pi x^2))^{-1/2} (\sec(\pi x^2) \tan(\pi x^2) (2\pi x)) \\ = \frac{\pi x \sec(\pi x^2) \tan(\pi x^2)}{\sqrt{6 + \sec(\pi x^2)}} \quad \square$$

$$\underline{237.} \quad y = \cot^3(4x+1)$$

$$\underline{\text{Sol:}} \quad \frac{dy}{dx} = 3 \cot^2(4x+1) (-\csc^2(4x+1) (4)) \\ = -12 \cot^2(4x+1) \csc^2(4x+1) \quad \square$$

238. Let $y = (f(x))^2$ and suppose $f'(1) = 4$ and $\frac{dy}{dx} = 10$ at $x=1$. Calculate $f(1)$.

Sol: $\frac{dy}{dx} = 2f(x) \cdot f'(x)$.

$$\therefore 10 = 2 \cdot f(1) \cdot f'(1) = 8f(1)$$

$$\therefore f(1) = \frac{5}{4} \quad \square$$

239. If $y = (f(x) + 5x^2)^4$, $f(-1) = -4$, and $\frac{dy}{dx} = 3$ when $x = -1$. Calculate $f'(-1)$.

Sol: $\frac{dy}{dx} = 4(f(x) + 5x^2)^3 (f'(x) + 10x)$

$$\therefore 3 = 4(f(-1) + 5(-1)^2)^3 (f'(-1) + 10(-1)) = 4(-4 + 5)^3 (f'(-1) - 10)$$

$$\therefore 3 = 4f'(-1) - 40, \quad \therefore f'(-1) = \frac{43}{4} \quad \square$$

240. Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

Sol: $y = (f(x^3 - 2x) + 3x)^2$

$$\frac{dy}{dx} = 2(f(x^3 - 2x) + 3x) (f'(x^3 - 2x)(3x^2 - 2) + 3)$$

$$\therefore 18 = 2(f(2^3 - 2 \cdot 2) + 3 \cdot 2) (f'(2^3 - 2 \cdot 2)(3 \cdot 2^2 - 2) + 3)$$

$$= 2(f(4) + 6)(f'(4) \cdot 10 + 3)$$

$$= 2(6 + 6)(10f'(4) + 3)$$

$$\therefore \frac{18}{24} = 10f'(4) + 3 \quad \therefore f'(4) = \left(\frac{3}{4} - 3\right) / 10 = \frac{-9}{40} \quad \square$$

241. Find an equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin.

Sol: $\frac{dy}{dx} = -\cos\left(\frac{x}{2}\right) \cdot \frac{1}{2}$,

so $m = -\cos\left(\frac{0}{2}\right) \cdot \frac{1}{2} = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$

$$y - 0 = -\frac{1}{2}(x - 0) \quad \text{OR} \quad y = -\frac{1}{2}x \quad \square$$

242. Find an equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at $(1, 16)$.

Sol: $\frac{dy}{dx} = 2\left(3x + \frac{1}{x}\right)\left(3 - x^{-2}\right)$,

so $m = 2\left(3 \cdot 1 + \frac{1}{1}\right)\left(3 - 1^{-2}\right) = 16$

$$y - 16 = 16(x - 1) \quad \text{OR} \quad y = 16x \quad \square$$

243. Find the x -coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

Sol: $\frac{dy}{dx} = 8\left(x - \frac{6}{x}\right)^7 \left(1 + \frac{6}{x^2}\right)$

so want $8\left(x - \frac{6}{x}\right)^7 \left(1 + \frac{6}{x^2}\right) = 0$

i.e. $x - \frac{6}{x} = 0$ or $1 + \frac{6}{x^2} = 0$

i.e. $x^2 - 6 = 0$ or $x + 6 = 0$

i.e. $x = \sqrt{6}$ or $x = -\sqrt{6}$ or $x = -6$. \square

244. Calculate the normal line to $g(\theta) = \sin^2(\pi\theta)$ at $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Sol: $g'(\theta) = 2\sin(\pi\theta) \cdot \cos(\pi\theta) \cdot \pi$.

$\therefore g'\left(\frac{1}{4}\right) = 2\pi \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) = 2\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \pi$.

\therefore Normal line has $m = -\frac{1}{\pi}$ and passes through $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$$y - \frac{1}{2} = -\frac{1}{\pi}\left(x - \frac{1}{4}\right) \quad \square$$

245-252: Use the table of values to calculate $h'(a)$ for the given $h(x)$ and a .

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	-2	3	0
2	4	4	1	-1
3	3	-3	2	3

245. $h(x) = f(g(x))$, $a = 0$.

Sol: $h'(x) = f'(g(x)) \cdot g'(x)$

$\therefore h'(0) = f'(g(0)) \cdot g'(0) = f'(0) \cdot 2 = 5 \cdot 2 = 10$ \square

246. $h(x) = g(f(x))$, $a = 0$.

Sol: $h'(x) = g'(f(x)) \cdot f'(x)$

$\therefore h'(0) = g'(f(0)) \cdot f'(0) = g'(2) \cdot 5 = -1 \cdot 5 = -5$ \square

247. $h(x) = (x^4 + g(x))^{-2}$, $a = 1$.

Sol: $h'(x) = -2(x^4 + g(x))^{-3} (4x^3 + g'(x))$

$\therefore h'(1) = -2(1^4 + g(1))(4 \cdot 1^3 + g'(1)) = -2(1+3)(4+0) = -32$ \square

248. $h(x) = \left(\frac{f(x)}{g(x)}\right)^2$, $a = 3$.

Sol: $h'(x) = 2 \left(\frac{f(x)}{g(x)}\right) \cdot \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{2f(x)(g(x)f'(x) - f(x)g'(x))}{(g(x))^3}$

$\therefore h'(3) = \frac{2f(3)(g(3)f'(3) - f(3)g'(3))}{(g(3))^3} = \frac{2 \cdot 3(2 \cdot 3 - 3 \cdot 3)}{2^3} = \frac{9}{4}$ \square

249. $h(x) = f(x + f(x))$, $a = 1$

Sol: $h'(x) = f'(x + f(x)) (1 + f'(x))$

$\therefore h'(1) = f'(1 + f(1)) (1 + f'(1)) = f'(1+1) (1 + (-2)) = f'(2) \cdot -1 = 4 \cdot -1 = -4$ \square

250. $h(x) = (1 + g(x))^3$, $a = 2$.

Sol: $h'(x) = 3(1 + g(x))^2 g'(x)$

$\therefore h'(2) = 3(1 + g(2))^2 g'(2) = 3(1+1)^2 (-1) = -12$ \square

251. $h(x) = g(2 + f(x^2))$, $a = 1$

Sol: $h'(x) = g'(2 + f(x^2)) f'(x^2) \cdot 2x$

$\therefore h'(1) = g'(2 + f(1^2)) f'(1^2) \cdot 2 \cdot 1 = g'(2 + f(1)) f'(1) \cdot 2 = g'(2+1) (-2) \cdot 2 = -4g'(3) = -4 \cdot 3 = -12$ \square

252. $h(x) = f(g(\sin(x)))$, $a = 0$

Sol: $h'(x) = f'(g(\sin(x))) \cdot g'(\sin(x)) \cdot \cos(x)$

$\therefore h'(0) = f'(g(\sin(0))) g'(\sin(0)) \cos(0) = f'(g(0)) g'(0) \cdot 1 = f'(0) \cdot 2 = 5 \cdot 2 = 10$ \square