# CHAIN RULE

214–219: Given $y=f(u)$ and $u=g(x)$, calculate $\frac{dy}{dx}$.

214. $y = 3u-6$, $u=2x^2$

Sol: $\frac{dy}{du}=3$, $\frac{du}{dx}=4x$, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=3\cdot 4x=12x$ $\boxed{\checkmark}$

215. $y=6u^3$, $u=7x-4$

Sol: $\frac{dy}{du}=6\cdot 3u^2=18u^2$, $\frac{du}{dx}=7$,

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=18(7x-4)^2\cdot 7=126(7x-4)^2$ $\boxed{\checkmark}$

216. $y=\sin(u)$, $u=5x-1$

Sol: $\frac{dy}{du}=\cos(u)$, $\frac{du}{dx}=5$, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos(5x-1)\cdot 5=5\cos(5x-1)$ $\boxed{\checkmark}$

217. $y=\cos(u)$, $u=\frac{-x}{8}=-\frac{1}{8}x$

Sol: $\frac{dy}{du}=-\sin(u)$, $\frac{du}{dx}=-\frac{1}{8}$, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=-\sin(-\frac{1}{8}x)\cdot -\frac{1}{8}=\frac{1}{8}\sin(-\frac{1}{8}x)$ $\boxed{\checkmark}$

218. $y=\tan(u)$, $u=9x+2$

Sol: $\frac{dy}{du}=\sec^2(u)$, $\frac{du}{dx}=9$, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\sec^2(9x+2)\cdot 9=9\sec^2(9x+2)$ $\boxed{\checkmark}$

219. $y=\sqrt{4u+3}$, $u=x^2-6x$

Sol: $y=(4u+3)^{1/2}$, $\frac{dy}{du}=\frac{1}{2}(4u+3)^{-1/2}\cdot 4=2(4u+3)^{-1/2}$, $\frac{du}{dx}=2x-6$

$\frac{dy}{dx}=2(4(x^2-6x)+3)^{-1/2}(2x-6)=\frac{4(x-3)}{\sqrt{4x^2-24x+3}}$ $\boxed{\checkmark}$

**220–227:** Decompose each $y(x)$ as $y=f(u)$ for $u=g(x)$, and take the derivative via the chain rule.

**220:** $y=(3x-2)^6$

**Sol:** $\left(\begin{array}{ll} y=u^6, & u=3x-2 \\ y'=6u^5, & u'=3 \end{array}\right. \rightsquigarrow \frac{dy}{dx}=6(3x-2)^5\cdot 3=18(3x-2)^5$ $\square$

**221:** $y=(3x^2+1)^3$

**Sol:** $\left(\begin{array}{ll} y=u^3 & u=3x^2+1 \\ y'=3u^2 & u'=6x \end{array}\right. \rightsquigarrow \frac{dy}{dx}=3(3x^2+1)^2\cdot 6x=18x(3x^2+1)^2$ $\square$

**222:** $y=\sin^5(x)$

**Sol:** $\left(\begin{array}{ll} y=u^5 & u=\sin(x) \\ y'=5u^4 & u'=\cos(x) \end{array}\right. \rightsquigarrow \frac{dy}{dx}=5(\sin(x))^4\cdot\cos(x)=5\sin^4(x)\cos(x)$ $\square$

**223.** $y=\left(\frac{x}{7}+\frac{7}{x}\right)^7$

**Sol:** $\left(\begin{array}{ll} y=u^7 & u=\frac{1}{7}x+7x^{-1} \\ y'=7u^6 & u'=\frac{1}{7}-7x^{-2} \end{array}\right. \rightsquigarrow \frac{dy}{dx}=7\left(\frac{1}{7}x+7x^{-1}\right)^6\left(\frac{1}{7}-7x^{-2}\right)=\left(1-7x^{-2}\right)\left(\frac{x}{7}+\frac{7}{x}\right)^6$ $\square$

**224.** $y=\tan(\sec(x))$

**Sol:** $\left(\begin{array}{ll} y=\tan(u) & u=\sec(x) \\ y'=\sec^2(u) & u'=\sec(x)\tan(x) \end{array}\right. \rightsquigarrow \frac{dy}{dx}=\sec^2(\sec(x))\cdot\sec(x)\tan(x)$ $\square$

**225.** $y=\csc(\pi x+1)$

**Sol:** $\left(\begin{array}{ll} y=\csc(u) & u=\pi x+1 \\ y'=-\csc(u)\cot(u) & u'=\pi \end{array}\right. \longrightarrow \frac{dy}{dx}=-\csc(\pi x+1)\cot(\pi x+1)\cdot\pi$ $\square$

**226.** $y=\cot^2(x)$

**Sol:** $\left(\begin{array}{ll} y=u^2 & u=\cot(x) \\ y'=2u & u'=-\csc^2(x) \end{array}\right. \longrightarrow \frac{dy}{dx}=2\cot(x)(-\csc^2(x))=-2\csc^2(x)\cot(x)$ $\square$

**227.** $y=-6(\sin(x))^{-3}$

**Sol:** $\left(\begin{array}{ll} y=-6u^{-3} & u=\sin(x) \\ y'=-6\cdot -3u^{-4}=18u^{-4} & u'=\cos(x) \end{array}\right. \rightsquigarrow \frac{dy}{dx}=18(\sin(x))^{-3}\cos(x)$ $\square$

228–237: Calculate $\frac{dy}{dx}$.

228. $y = (3x^2+3x-1)^4$

Sol: $\frac{dy}{dx} = 4(3x^2+3x-1)^3(6x+3)$

$= 12(2x+1)(3x^2+3x-1)^3$ ◪

229. $y = (5-2x)^{-2}$

Sol: $\frac{dy}{dx} = -2(5-2x)^{-3}(-2)$

$= 4(5-2x)^{-3}$ ◪

230. $y = \cos^3(\pi x)$

Sol: $\frac{dy}{dx} = 3\cos^2(\pi x)\cdot(-\sin(\pi x)\cdot\pi)$

$= -3\pi\cos^2(\pi x)\sin(\pi x)$ ◪

231. $y = (2x^3-x^2+6x+1)^3$

Sol: $\frac{dy}{dx} = 3(2x^3-x^2+6x+1)^2(6x^2-2x+6)$

$= 6(3x^2-x+3)(2x^3-x^2+6x+1)^2$ ◪

232. $y = \frac{1}{\sin^2(x)} = (\sin(x))^{-2}$

Sol: $\frac{dy}{dx} = -2(\sin(x))^{-3}\cos(x)$

$= \frac{-2\cos(x)}{\sin^3(x)}$ ◪

233. $y = \left(\tan(x) + \sin(x)\right)^{-3}$

Sol: $\frac{dy}{dx} = -3\left(\tan(x) + \sin(x)\right)^{-4}\left(\sec^2(x) + \cos(x)\right)$ $\boxed{/}$

234: $y = x^2 \cos^4(x)$

Sol: $\frac{dy}{dx} = 2x\cos^4(x) + x^2\left(4\cos^3(x)\left(-\sin(x)\right)\right)$

$= 2x\cos^3(x)\left(\cos(x) - 2x\sin(x)\right)$ $\boxed{/}$

235. $y = \sin(\cos(7x))$

Sol: $\frac{dy}{dx} = \cos(\cos(7x))\left(-\sin(7x) \cdot 7\right)$

$= -7\cos(\cos(7x))\sin(7x)$ $\boxed{/}$

236. $y = \sqrt{6 + \sec(\pi x^2)} = \left(6 + \sec(\pi x^2)\right)^{1/2}$

Sol: $\frac{dy}{dx} = \frac{1}{2}\left(6 + \sec(\pi x^2)\right)^{-1/2}\left(\sec(\pi x^2)\tan(\pi x^2)\ (2\pi x)\right)$

$= \frac{\pi x \sec(\pi x^2)\tan(\pi x^2)}{\sqrt{6 + \sec(\pi x^2)}}$ $\boxed{/}$

237. $y = \cot^3(4x+1)$

Sol: $\frac{dy}{dx} = 3\cot^2(4x+1)\left(-\csc^2(4x+1)\ (4)\right)$

$= -12\cot^2(4x+1)\csc^2(4x+1)$ $\boxed{/}$

238. Let $y = (f(x))^2$ and suppose $f'(1) = 4$ and $\frac{dy}{dx} = 10$ at $x = 1$. Calculate $f(1)$.

Sol: $\frac{dy}{dx} = 2f(x) \cdot f'(x)$.

$\therefore 10 = 2 \cdot f(1) \cdot f'(1) = 8 f(1)$

$\therefore f(1) = \frac{5}{4}$ ▨

239. If $y = (f(x) + 5x^2)^4$, $f(-1) = -4$, and $\frac{dy}{dx} = 3$ when $x = -1$. Calculate $f'(-1)$.

Sol: $\frac{dy}{dx} = 4(f(x) + 5x^2)^3 (f'(x) + 10x)$

$\therefore 3 = 4(f(-1) + 5(-1)^2)^3 (f'(-1) + 10(-1)) = 4(-4+5)^3 (f'(-1) - 10)$

$\therefore 3 = 4f'(-1) - 40$, $\therefore f'(-1) = \frac{43}{4}$ ▨

240. Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

Sol: $y = (f(x^3 - 2x) + 3x)^2$

$$\frac{dy}{dx} = 2(f(x^3 - 2x) + 3x)(f'(x^3 - 2x)(3x^2 - 2) + 3)$$

$$\therefore 18 = 2(f(2^3 - 2 \cdot 2) + 3 \cdot 2)(f'(2^3 - 2 \cdot 2)(3 \cdot 2^2 - 2) + 3)$$

$$= 2(f(4) + 6)(f'(4) \cdot 10 + 3)$$

$$= 2(6 + 6)(10f'(4) + 3)$$

$\therefore \frac{18}{24} = 10f'(4) + 3$ $\therefore f'(4) = \left(\frac{3}{4} - 3\right)/10 = \frac{-9}{40}$ ▨

241. Find an equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin.

Sol: $\frac{dy}{dx} = -\cos\left(\frac{x}{2}\right)\cdot\frac{1}{2}$,

so $m = -\cos\left(\frac{0}{2}\right)\cdot\frac{1}{2} = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$

$y - 0 = -\frac{1}{2}(x-0)$ OR $y = -\frac{1}{2}x$ $\boxed{/\!/}$

242. Find an equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at $(1, 16)$.

Sol: $\frac{dy}{dx} = 2\left(3x + \frac{1}{x}\right)\left(3 - x^{-2}\right)$,

so $m = 2\left(3\cdot 1 + \frac{1}{1}\right)\left(3 - 1^{-2}\right) = 16$

$y - 16 = 16(x-1)$ OR $y = 16x$ $\boxed{/\!/}$

243. Find the x-coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

Sol: $\frac{dy}{dx} = 8\left(x - \frac{6}{x}\right)^7\left(1 + \frac{6}{x^2}\right)$

So want $8\left(x - \frac{6}{x}\right)^7\left(1 + \frac{6}{x^2}\right) = 0$

i.e. $x - \frac{6}{x} = 0$ or $1 + \frac{6}{x} = 0$

i.e. $x^2 - 6 = 0$ or $x + 6 = 0$

i.e. $x = \sqrt{6}$ or $x = -\sqrt{6}$ or $x = -6$. $\boxed{/\!/}$

244. Calculate the normal line to $g(\theta) = \sin^2(\pi\theta)$ at $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Sol: $g'(\theta) = 2\sin(\pi\theta)\cdot\cos(\pi\theta)\cdot\pi$.

$\therefore g'\left(\frac{1}{4}\right) = 2\pi\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 2\pi\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \pi$.

$\therefore$ Normal line has $m = -\frac{1}{\pi}$ and passes through $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$y - \frac{1}{2} = -\frac{1}{\pi}\left(x - \frac{1}{4}\right)$ $\boxed{/\!/}$

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 2 | 5 | 0 | 2 |
| 1 | 1 | −2 | 3 | 0 |
| 2 | 4 | 4 | 1 | −1 |
| 3 | 3 | −3 | 2 | 3 |

**245–252**: Use the table of values to calculate $h'(a)$ for the given $h(x)$ and $a$.

**245.** $h(x) = f(g(x))$, $a = 0$.

Sol: $h'(x) = f'(g(x)) \cdot g'(x)$

$\therefore h'(0) = f'(g(0)) \cdot g'(0) = f'(0) \cdot 2 = 5 \cdot 2 = 10$

**246.** $h(x) = g(f(x))$, $a = 0$.

Sol: $h'(x) = g'(f(x)) \cdot f'(x)$

$\therefore h'(0) = g'(f(0)) \cdot f'(0) = g'(2) \cdot 5 = -1 \cdot 5 = -5$

**247.** $h(x) = (x^4 + g(x))^{-2}$, $a = 1$.

Sol: $h'(x) = -2(x^4 + g(x))^{-3}(4x^3 + g'(x))$

$\therefore h'(1) = -2(1^4 + g(1))(4 \cdot 1^3 + g'(1)) = -2(1+3)(4+0) = -32$

**248.** $h(x) = \left(\frac{f(x)}{g(x)}\right)^2$, $a = 3$.

Sol: $h'(x) = 2\left(\frac{f(x)}{g(x)}\right) \cdot \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{2f(x)\left(g(x)f'(x) - f(x)g'(x)\right)}{(g(x))^3}$

$\therefore h'(3) = \frac{2f(3)(g(3)f'(3) - f(3)g'(3))}{(g(3))^3} = \frac{2 \cdot 3(2 \cdot -3 - 3 \cdot 3)}{2^3} = \frac{9}{4}$

**249.** $h(x) = f(x + f(x))$, $a = 1$

Sol: $h'(x) = f'(x + f(x))(1 + f'(x))$

$\therefore h'(1) = f'(1 + f(1))(1 + f'(1)) = f'(1+1)(1 + (-2)) = f'(2) \cdot -1 = 4 \cdot -1 = -4$

**250.** $h(x) = (1 + g(x))^3$, $a = 2$.

Sol: $h'(x) = 3(1 + g(x))^2 g'(x)$

$\therefore h'(2) = 3(1 + g(2))^2 \cdot g'(2) = 3(1+1)^2(-1) = -12$

**251.** $h(x) = g(2 + f(x^2))$, $a = 1$

Sol: $h'(x) = g'(2 + f(x^2)) f'(x^2) \cdot 2x$

$\therefore h'(1) = g'(2 + f(1^2)) f'(1^2) \cdot 2 \cdot 1 = g'(2 + f(1)) f'(1) \cdot 2 = g'(2+1)(-2)2 = -4g'(3) = -4 \cdot 3 = -12$

**252.** $h(x) = f(g(\sin(x)))$, $a = 0$

Sol: $h'(x) = f'(g(\sin(x))) \cdot g'(\sin(x)) \cdot \cos(x)$

$\therefore h'(0) = f'(g(\sin(0)))g'(\sin(0))\cos(0) = f'(g(0))g'(0) \cdot 1 = f'(0) \cdot 2 = 5 \cdot 2 = 10$