Derivative Practice

1. Calculate the derivative $f'(x)$ for...

1a. $f(x) = \dfrac{x^3 - 6x}{1+2x^2}$

Sol: $f'(x) = \dfrac{d}{dx}\left[\dfrac{x^3-6x}{1+2x^2}\right] = \dfrac{(1+2x^2)\frac{d}{dx}[x^3-6x] - (x^3-6x)\frac{d}{dx}[1+2x^2]}{(1+2x^2)^2}$

$$= \frac{(1+2x^2)(3x^2-6) - (x^3-6x)(4x)}{(1+2x^2)^2}$$

$$= \frac{3x^2 - 6 + 6x^4 - 12x^2 - 4x^4 + 24x^2}{(1+2x^2)^2}$$

$$= \frac{2x^4 + 15x^2 - 6}{(1+2x^2)^2} \quad ▨$$

1b. $f(x) = x^2\cos(x) - x\sin(x)$

Sol: $f'(x) = \dfrac{d}{dx}\left[x^2\cos(x) - x\sin(x)\right] = \dfrac{d}{dx}\left[x^2\cos(x)\right] - \dfrac{d}{dx}\left[x\sin(x)\right]$

$$= \left(\frac{d}{dx}[x^2]\cos(x) + x^2\frac{d}{dx}[\cos(x)]\right) - \left(\frac{d}{dx}[x]\sin(x) + x\frac{d}{dx}[\sin(x)]\right)$$

$$= 2x\cos(x) + x^2(-\sin(x)) - \sin(x) - x\cos(x)$$

$$= x\cos(x) - (x^2+1)\sin(x) \quad ▨$$

1c. $f(x) = \dfrac{x^2-1}{\cos(x)}$

Sol: $f'(x) = \dfrac{d}{dx}\left[\dfrac{x^2-1}{\cos(x)}\right] = \dfrac{\cos(x)\frac{d}{dx}[x^2-1] - (x^2-1)\frac{d}{dx}[\cos(x)]}{\cos^2(x)}$

$$= \frac{\cos(x)\cdot 2x - (x^2-1)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{2x\cos(x) + (x^2-1)\sin(x)}{\cos^2(x)} \quad ▨$$

2. Find an equation to the tangent line to $y=f(x)$ at $x=a$.

2a. $y = x^2 + \frac{15}{x} - 10, \quad a = 5$

Sol: $y - y_0 = m(x - x_0)$ ← equation of a line

$y'(x) = 2x - 15x^{-2} = 2x - \frac{15}{x^2}$

$m = y'(5) = 2\cdot 5 - \frac{15}{5^2} = 10 - \frac{3}{5} = \frac{47}{5}$

$x_0 = 5, \quad y_0 = y(5) = 5^2 + \frac{15}{5} - 10 = 18$

$y - 18 = \frac{47}{5}(x - 5)$ ☒

2b. $y = (3x - x^2)(3 - x - x^2), \quad a = 1$

Sol: $y - y_0 = m(x - x_0)$ ← equation of a line

$y'(x) = \frac{d}{dx}[3x - x^2](3 - x - x^2) + (3x - x^2)\frac{d}{dx}[3 - x - x^2]$

$= (3 - 2x)(3 - x - x^2) + (3x - x^2)(-1 - 2x)$

$m = y'(1) = (3 - 2\cdot 1)(3 - 1 - 1^2) + (3\cdot 1 - 1^2)(-1 - 2\cdot 1) = 1 + 2(-3) = -5$

$x_0 = 1, \quad y_0 = y(1) = (3\cdot 1 - 1^2)(3 - 1 - 1^2) = 2\cdot 1 = 2$

$y - 2 = -5(x - 1)$ ☒

3. Use the quotient rule and trigonometric identities to show...

3a. $\frac{d}{dx}[\tan(x)] = \sec^2(x)$

Sol: $\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\frac{d}{dx}[\sin(x)] - \sin(x)\frac{d}{dx}[\cos(x)]}{(\cos(x))^2}$

$= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$

$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$ ▨

3b. $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

Sol: $\frac{d}{dx}[\sec(x)] = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{\cos(x)\frac{d}{dx}[1] - 1\cdot\frac{d}{dx}[\cos(x)]}{(\cos(x))^2}$

$= \frac{\cos(x)\cdot 0 - (-\sin(x))}{\cos^2(x)}$

$= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$ ▨

3c. $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

Sol: $\frac{d}{dx}[\csc(x)] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{\sin(x)\cdot\frac{d}{dx}[1] - 1\cdot\frac{d}{dx}[\sin(x)]}{(\sin(x))^2}$

$= \frac{\sin(x)\cdot 0 - \cos(x)}{\sin^2(x)}$

$= -\frac{\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$ ▨

3d. $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$

Sol: $\frac{d}{dx}[\cot(x)] = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] = \frac{\sin(x)\frac{d}{dx}[\cos(x)] - \cos(x)\frac{d}{dx}[\sin(x)]}{(\sin(x))^2}$

$= \frac{\sin(x)\cdot(-\sin(x)) - \cos(x)\cdot\cos(x)}{\sin^2(x)}$

$= -\left(\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)}\right)$

$= -\frac{1}{\sin^2(x)} = -\csc^2(x)$ ▨

4. Calculate the indicated higher derivative.

4a. $\frac{d^2}{dx^2}\left[\frac{\cos(x)}{x}\right]$ $\quad y=\frac{\cos(x)}{x}$

Sol: $\frac{dy}{dx}=\frac{d}{dx}\left[\frac{\cos(x)}{x}\right]=\frac{x\frac{d}{dx}[\cos(x)]-\cos(x)\frac{d}{dx}[x]}{x^2}=\frac{-x\sin(x)-\cos(x)}{x^2}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-x\sin(x)-\cos(x)}{x^2}\right]=\frac{x^2\frac{d}{dx}[-x\sin(x)-\cos(x)]-(-x\sin(x)-\cos(x))\frac{d}{dx}[x^2]}{(x^2)^2}$$

$$=\frac{x^2\left(\frac{d}{dx}[-x]\sin(x)+(-x)\frac{d}{dx}[\sin(x)]-(-\sin(x)\right)+(x\sin(x)+\cos(x))(2x)}{x^4}$$

$$=\frac{x^2(-\sin(x)-x\cos(x)-\cos(x))+x(2x\sin(x)+2\cos(x))}{x^4}$$

$$=\frac{x(-x\sin(x)-x^2\cos(x)-x\cos(x)+2x\sin(x)+2\cos(x))}{x^4}$$

$$=\frac{(-x^2-x+2)\cos(x)+x\sin(x)}{x^3}$$ ▨

4b. $\frac{d^3}{dx^3}[x^2+3x-7]$ $\quad y=x^2+3x-7$

Sol: $\frac{dy}{dx}=2x+3,\quad \frac{d^2y}{dx^2}=2,\quad \frac{d^3y}{dx^3}=0$ ▨

4c. $\frac{d^2}{dx^2}[\sec(x)]$ $\quad y=\sec(x)$

Sol: $\frac{dy}{dx}=\frac{d}{dx}[\sec(x)]=\sec(x)\tan(x)$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[\sec(x)\tan(x)]=\frac{d}{dx}[\sec(x)]\tan(x)+\sec(x)\frac{d}{dx}[\tan(x)]$$

$$=\sec(x)\tan(x)\cdot\tan(x)+\sec(x)\cdot\sec^2(x)=\sec(x)\left(\tan^2(x)+\sec^2(x)\right)$$

4d: $\frac{d^2}{dx^2}[x\sin(x)]$ $\quad y=x\sin(x)$

Sol: $\frac{dy}{dx}=\frac{d}{dx}[x]\sin(x)+x\frac{d}{dx}[\sin(x)]=\sin(x)+x\cos(x)$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[\sin(x)+x\cos(x)]=\cos(x)+\frac{d}{dx}[x]\cos(x)+x\frac{d}{dx}[\cos(x)]$$

$$=2\cos(x)-x\sin(x)$$ ▨