The Derivative as a Function

54-63: Use the limit definition of the derivative to find f'.

54. $f(x) = 6$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{6-6}{h} = \lim_{h\to 0} 0 = 0.$$

$\therefore f'(x) = 0$

55. $f(x) = 2-3x$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(2-3(x+h)) - (2-3x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h-x)}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

$\therefore f'(x) = 3$

56. $f(x) = \frac{2}{7}x + 1$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\frac{2}{7}(x+h)+1) - (\frac{2}{7}x+1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2}{7}(x+h-x)}{h} = \lim_{h\to 0} \frac{2}{7}\cdot\frac{h}{h} = \lim_{h\to 0} \frac{2}{7} = \frac{2}{7}$$

$\therefore f'(x) = \frac{2}{7}$

57. $f(x) = 4x^2$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h\to 0} \frac{4(x+h+x)(x+h-x)}{h} = \lim_{h\to 0} \frac{4(2x+h)h}{h}$$

$$= \lim_{h\to 0} 4(2x+h) = 4(2x+0) = 8x.$$

$\therefore f'(x) = 8x$

58. $f(x) = 5x - x^2$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(5(x+h)-(x+h)^2) - (5x-x^2)}{h}$$

$$= \lim_{h\to 0} \frac{5(x+h-x) - ((x+h)^2 - x^2)}{h}$$

$$= \lim_{h\to 0} \frac{5h - (x+h+x)(x+h-x)}{h}$$

$$= \lim_{h\to 0} \frac{h(5-2x-h)}{h}$$

$$= \lim_{h\to 0} (5-2x-h) = 5-2x-0 = 5-2x$$

$\therefore f'(x) = 5-2x$

59. $f(x)=\sqrt{2x}$

Sol: $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\cdot\frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}$$

$$=\lim_{h\to0}\frac{2(x+h)-2x}{h\left(\sqrt{2(x+h)}+\sqrt{2x}\right)}$$

$$=\lim_{h\to0}\frac{2h}{h\left(\sqrt{2x+2h}+\sqrt{2x}\right)}$$

$$=\lim_{h\to0}\frac{2}{\sqrt{2x+2h}+\sqrt{2x}}=\frac{2}{\sqrt{2x+2\cdot0}+\sqrt{2x}}=\frac{2}{2\sqrt{x}}=\frac{1}{\sqrt{x}}$$

$$\therefore\ f'(x)=\frac{1}{\sqrt{x}}$$

60. $f(x)=\sqrt{x-6}$

Sol: $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\sqrt{(x+h)-6}-\sqrt{x-6}}{h}\cdot\frac{\sqrt{x+h-6}+\sqrt{x-6}}{\sqrt{x+h-6}+\sqrt{x-6}}$$

$$=\lim_{h\to0}\frac{(x+h-6)-(x-6)}{h\left(\sqrt{x+h-6}+\sqrt{x-6}\right)}=\lim_{h\to0}\frac{h}{h\left(\sqrt{x+h-6}+\sqrt{x-6}\right)}$$

$$=\lim_{h\to0}\frac{1}{\sqrt{x+h-6}+\sqrt{x-6}}=\frac{1}{\sqrt{x+0-6}+\sqrt{x-6}}=\frac{1}{2\sqrt{x-6}}$$

$$\therefore\ f'(x)=\frac{1}{2\sqrt{x-6}}$$

61. $f(x)=\frac{9}{x}$

Sol: $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\frac{9}{x+h}-\frac{9}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)}$$

$$=\lim_{h\to0}\frac{9x-9(x+h)}{hx(x+h)}=\lim_{h\to0}\frac{-9h}{hx(x+h)}$$

$$=\lim_{h\to0}\frac{-9}{x(x+h)}=-\frac{9}{x(x+0)}=-\frac{9}{x^2}.$$

$$\therefore\ f'(x)=-\frac{9}{x^2}$$

62. $f(x)=x+\frac{1}{x}$

Sol: $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\left((x+h)+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h}$$

$$=\lim_{h\to0}\frac{(x+h-x)+\left(\frac{1}{x+h}-\frac{1}{x}\right)}{h}$$

$$=\lim_{h\to0}\left(\frac{h}{h}+\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)}\right)$$

$$=\lim_{h\to0}\left(1+\frac{x-(x+h)}{hx(x+h)}\right)$$

$$=\lim_{h\to0}\left(1-\frac{h}{hx(x+h)}\right)$$

$$=\lim_{h\to0}\left(1-\frac{1}{x(x+h)}\right)=1-\frac{1}{x(x+0)}=1-\frac{1}{x^2}$$

$$\therefore\ f'(x)=1-\frac{1}{x^2}$$

63. $f(x) = \frac{1}{\sqrt{x}}$

Sol: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x+0}(\sqrt{x} + \sqrt{x+0})} = \frac{-1}{2x\sqrt{x}}$$

$\therefore f'(x) = \frac{-1}{2x\sqrt{x}}$ ☑

68–73: Express the given limit as $f'(a)$ for some function $f(x)$ and input a.

Note: to check, start with $f(x)$ and a as given, and write out $f'(a)$ using the limit definition :)

68. $\lim\limits_{h \to 0} \frac{(1+h)^{2/3} - 1}{h}$

Sol: $\lim\limits_{h \to 0} \frac{(1+h)^{2/3} - 1}{h} = \lim\limits_{h \to 0} \frac{(1+h)^{2/3} - 1^{2/3}}{h}$.

$\therefore f(x) = x^{2/3}$ and $a = 1$ ◪

69. $\lim\limits_{h \to 0} \frac{(3(2+h)^2 + 2) - 14}{h}$

Sol: $\lim\limits_{h \to 0} \frac{(3(2+h)^2 + 2) - 14}{h} = \lim\limits_{h \to 0} \frac{(3(2+h)^2 + 2) - (3 \cdot 2^2 + 2)}{h}$

$\therefore f(x) = 3x^2 + 2$ and $a = 2$ ◪

70. $\lim\limits_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

Sol: $\lim\limits_{h \to 0} \frac{\cos(\pi + h) + 1}{h} = \lim\limits_{h \to 0} \frac{\cos(\pi + h) - (-1)}{h} = \lim\limits_{h \to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h}$

$\therefore f(x) = \cos(x)$ and $a = \pi$ ◪

71. $\lim\limits_{h \to 0} \frac{(2+h)^4 - 16}{h}$

Sol: $\lim\limits_{h \to 0} \frac{(2+h)^4 - 16}{h} = \lim\limits_{h \to 0} \frac{(2+h)^4 - 2^4}{h}$

$\therefore f(x) = x^4$ and $a = 2$ ◪

72. $\lim\limits_{h \to 0} \frac{(2(3+h)^2 - (3+h)) - 15}{h}$

Sol: $\lim\limits_{h \to 0} \frac{(2(3+h)^2 - (3+h)) - 15}{h} = \lim\limits_{h \to 0} \frac{(2(3+h)^2 - (3+h)) - (2 \cdot 3^2 - 3)}{h}$

$\therefore f(x) = 2x^2 - x$ and $a = 3$ ◪

73. $\lim\limits_{h \to 0} \frac{e^h - 1}{h}$

Sol: $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = \lim\limits_{h \to 0} \frac{e^{0+h} - e^0}{h}$

$\therefore f(x) = e^x$ and $a = 0$ ◪

74–77: Show $f(x)$ is not differentiable at $x=1$.

74. $f(x) = \begin{cases} 2\sqrt{x} & \text{if } 0 \le x \le 1 \\ 3x-1 & \text{if } x > 1 \end{cases}$

Sol: We show $\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} \ne \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$, so $f'(1) = \lim_{x\to1} \frac{f(x)-f(1)}{x-1}$ DNE.

$$\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{2\sqrt{x}-2\sqrt{1}}{x-1}$$

$$= \lim_{x\to1^-} \frac{2(\sqrt{x}-1)}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to1^-} \frac{2(x-1)}{(x-1)(\sqrt{x}+1)} = \lim_{x\to1^-} \frac{2}{\sqrt{x}+1} = \frac{2}{\sqrt{1}+1} = 1$$

$$\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{(3x-1)-2\sqrt{1}}{x-1}$$

$$= \lim_{x\to1^+} \frac{3(x-1)}{x-1} = \lim_{x\to1^+} 3 = 3$$

$\therefore \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = 1 \ne 3 = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$, so $f'(1)$ DNE. ☒

75. $f(x) = \begin{cases} 3 & \text{if } x < 1 \\ 3x & \text{if } x \ge 1 \end{cases}$

Sol: Again, we show $\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} \ne \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$.

$$\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{3-3\cdot1}{x-1} = \lim_{x\to1^-} 0 = 0$$

$$\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{3x-3\cdot1}{x-1} = \lim_{x\to1^+} \frac{3(x-1)}{x-1} = \lim_{x\to1^+} 3 = 3$$

$\therefore \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = 0 \ne 3 = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$, so $f'(1)$ DNE. ☒

76. $f(x) = \begin{cases} -x^2+2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$

Sol: We show $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \ne \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{(-x^2+2)-(-1^2+2)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{-(x^2-1^2)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^-} (x+1) = 1+1 = 2$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{x-(-1^2+2)}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = \lim_{x\to 1^+} 1 = 1$$

$\therefore \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = 2 \ne 1 = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$, So $f'(1)$ DNE ☒

77. $f(x) = \begin{cases} 2x & \text{if } x \le 1 \\ \frac{2}{x} & \text{if } x > 1 \end{cases}$

Sol: We show $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \ne \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{2x-2\cdot 1}{x-1} = \lim_{x\to 1^-} \frac{2(x-1)}{x-1} = \lim_{x\to 1^-} 2 = 2$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{2}{x}-2\cdot 1}{x-1} \cdot \frac{x}{x}$$

$$= \lim_{x\to 1^+} \frac{2-2x}{x(x-1)}$$

$$= \lim_{x\to 1^+} \frac{-2(x-1)}{x(x-1)} = \lim_{x\to 1^+} -\frac{2}{x} = -\frac{2}{1} = -2$$

$\therefore \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = 2 \ne -2 = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$, So $f'(1)$ DNE ☒

81 – 83: Calculate $f''(x)$.

81. $f(x) = 2-3x$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(2-3(x+h)) - (2-3x)}{h}$$
$$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{3-3}{h} = \lim_{h\to 0} 0 = 0$$

82. $f(x) = 4x^2$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4(x+h)^2-4x^2}{h} = \lim_{h\to 0} \frac{4(x+h-x)(x+h+x)}{h}$$
$$= \lim_{h\to 0} \frac{4h(2x+h)}{h} = \lim_{h\to 0} 4(2x+h) = 4(2x+0) = 8x.$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{8(x+h)-8x}{h} = \lim_{h\to 0} \frac{8h}{h} = \lim_{h\to 0} 8 = 8.$$

83. $f(x) = x + \frac{1}{x}$

Sol: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left((x+h) + \frac{1}{x+h}\right) - \left(x + \frac{1}{x}\right)}{h}$$
$$= \lim_{h\to 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$
$$= \lim_{h\to 0} \frac{hx(x+h) + x - (x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{h\,(x(x+h)-1)}{hx(x+h)}$$
$$= \lim_{h\to 0} \frac{x(x+h)-1}{x(x+h)} = \frac{x(x+0)-1}{x(x+0)} = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{\left(1-\frac{1}{(x+h)^2}\right) - \left(1-\frac{1}{x^2}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-1}{(x+h)^2} + \frac{1}{x^2}}{h} \cdot \frac{x^2(x+h)^2}{x^2(x+h)^2} = \lim_{h\to 0} \frac{-x^2+(x+h)^2}{hx^2(x+h)^2}$$
$$= \lim_{h\to 0} \frac{(x+h)^2-x^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{(x+h-x)(x+h+x)}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{h(2x+h)}{hx^2(x+h)^2}$$
$$= \lim_{h\to 0} \frac{2x+h}{x^2(x+h)^2} = \frac{2x+0}{x^2(x+0)^2} = \frac{2x}{x^4} = \frac{2}{x^3}$$