Homework 5

62. Calculate $f'(x)$ for $f(x) = x + \frac{1}{x}$ via the definition.

Sol. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left((x+h)+\frac{1}{x+h}\right) - \left(x+\frac{1}{x}\right)}{h}$

$= \lim_{h\to 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0}\left(1 + \frac{1}{h}\cdot\left(\frac{1}{x+h}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{x+h}{x+h}\right)\right)$

$= \lim_{h\to 0}\left(1 + \frac{1}{h}\cdot\frac{x-(x+h)}{x(x+h)}\right)$

$= \lim_{h\to 0}\left(1+\frac{1}{h}\cdot\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\left(1 - \frac{1}{x(x+h)}\right)$

$= 1 - \frac{1}{x(x+0)} = 1 - \frac{1}{x^2}$ ☑

72. What are $f(x)$ and a if $f'(a) = \lim_{h\to 0} \frac{\left(2(3+h)^2 - (3+h)\right) - 15}{h}$?

Sol: $f(x) = 2x^2 - x$ and $a = 3$.

Check: $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$\therefore f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$= \lim_{h\to 0}\frac{\left(2(3+h)^2-(3+h)\right)-\left(2\cdot 3^2 - 3\right)}{h}$

$= \lim_{h\to 0}\frac{\left(2(3+h)^2-(3+h)\right)-(2\cdot 9-3)}{h}$

$= \lim_{h\to 0}\frac{\left(2(3+h)^2-(3+h)\right)-15}{h}$ ✓ ☑

76. Show $f'(1)$ does not exist for $f(x) = \begin{cases} -x^2+2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$

Sol: We'll calculate the limit corresponding to $f'(1)$ from both sides.

Note $f(1) = -1^2+2 = 1$.

$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{(-x^2+2)-1}{x-1} = \lim_{x\to 1^-}\frac{-(x^2-1)}{x-1} = \lim_{x\to 1^-}\frac{-(x-1)(x+1)}{x-1} = \lim_{x\to 1^-} -(x+1) = -(1+1) = -2$

$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{x-1}{x-1} = \lim_{x\to 1^+} 1 = 1.$

$\therefore \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = -2 \neq 1 = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$, so $f'(1)$ does not exist! ☑

83. Calculate $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ for $f(x) = x + \frac{1}{x}$.

Sol. From problem 62 we know $f'(x) = 1 - \frac{1}{x^2}$.

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{\left(1 - \frac{1}{(x+h)^2}\right) - \left(1 - \frac{1}{x^2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-\frac{1}{(x+h)^2} + \frac{1}{x^2}}{h} = \lim_{h \to 0} \left(\frac{1}{hx^2} - \frac{1}{h(x+h)^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{(x+h)^2}{hx^2(x+h)^2} - \frac{x^2}{hx^2(x+h)^2} \right) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{h(2x+h)}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{2x+h}{x^2(x+h)^2} = \frac{2x+0}{x^2(x+0)^2}$$

$$= \frac{2x}{x^4} = \frac{2}{x^3}$$

$$\therefore f''(x) = \frac{2}{x^3} \quad ▨$$