

# HOMEWORK 5

62. Calculate  $f'(x)$  for  $f(x) = x + \frac{1}{x}$  via the definition.

Sol.  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h) + \frac{1}{x+h} - (x + \frac{1}{x})}{h}$

$$= \lim_{h \rightarrow 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \left( 1 + \frac{1}{h} \cdot \left( \frac{1}{x+h} \cdot x - \frac{1}{x} \cdot \frac{x+h}{x+h} \right) \right)$$
$$= \lim_{h \rightarrow 0} \left( 1 + \frac{1}{h} \cdot \frac{x - (x+h)}{x(x+h)} \right)$$
$$= \lim_{h \rightarrow 0} \left( 1 + \frac{1}{h} \cdot \frac{-h}{x(x+h)} \right) = \lim_{h \rightarrow 0} \left( 1 - \frac{1}{x(x+h)} \right)$$
$$= 1 - \frac{1}{x(x+0)} = 1 - \frac{1}{x^2} \quad \square$$

72. What are  $f(x)$  and  $a$  if  $f'(a) = \lim_{h \rightarrow 0} \frac{(2(3+h)^2 - (3+h)) - 15}{h}$ ?

Sol:  $f(x) = 2x^2 - x$  and  $a = 3$ .

Check:  $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$\therefore f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2(3+h)^2 - (3+h)) - (2 \cdot 3^2 - 3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2(3+h)^2 - (3+h)) - (2 \cdot 9 - 3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2(3+h)^2 - (3+h)) - 15}{h} \quad \checkmark \quad \square$$

76. Show  $f'(1)$  does not exist for  $f(x) = \begin{cases} -x^2 + 2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$

Sol: We'll calculate the limit corresponding to  $f'(1)$  from both sides.

Note  $f(1) = -1^2 + 2 = 1$ .

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(-x^2 + 2) - 1}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x^2 - 1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x-1)(x+1)}{x-1} = \lim_{x \rightarrow 1^-} -(x+1) = -(1+1) = -2$$

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1^+} 1 = 1$$

$\therefore \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = -2 \neq 1 = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$ , so  $f'(1)$  does not exist!  $\square$

83. Calculate  $f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$  for  $f(x) = x + \frac{1}{x}$ .

Sol. From problem 62 we know  $f'(x) = 1 - \frac{1}{x^2}$ .

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{\left(1 - \frac{1}{(x+h)^2}\right) - \left(1 - \frac{1}{x^2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-\frac{1}{(x+h)^2} + \frac{1}{x^2}}{h} = \lim_{h \rightarrow 0} \left( \frac{1}{hx^2} - \frac{1}{h(x+h)^2} \right)$$

$$= \lim_{h \rightarrow 0} \left( \frac{(x+h)^2}{hx^2(x+h)^2} - \frac{x^2}{hx^2(x+h)^2} \right) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{hx^2(x+h)^2}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{hx^2(x+h)^2} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{hx^2(x+h)^2}$$

$$= \lim_{h \rightarrow 0} \frac{2x+h}{x^2(x+h)^2} = \frac{2x+0}{x^2(x+0)^2}$$

$$= \frac{2x}{x^4} = \frac{2}{x^3}$$

$$\therefore f''(x) = \frac{2}{x^3} \quad \square$$