CONTINUITY

131-138: Find all points of discontinuity.

131: $f(x) = \frac{1}{\sqrt{x}}$ ← Continuous on its domain.

$\sqrt{x} = 0 \iff x = 0^2 = 0$ ← division by 0.

$x \geqslant 0$ ← Square root of negative.

∴ f is cts on $(0, \infty)$ ☑

132. $f(x) = \frac{2}{x^2+1}$ ← cts on domain

$\quad x^2+1 = 0 \iff x^2 = -1$ ← division by zero.
$\qquad\qquad\qquad\quad \underset{\text{Reject!}}{\uparrow}$

∴ f is cts on $(-\infty, \infty)$. ☑

133. $f(x) = \frac{x}{x^2-x}$ cts on domain.

$x^2-x = 0 \iff x(x-1) = 0$

$\qquad\qquad \iff x=0 \text{ or } x-1=0$
$\qquad\qquad \iff x=0 \text{ or } x=1$ ← Division by zero.

∴ f is cts on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ ☑

Follow-up to 133: Why didn't I write this:

$\quad f(x) = \frac{x}{x^2-x} = \frac{x}{x(x-1)} = \frac{1}{x-1}$, so cts on $(-\infty, 1) \cup (1, \infty)$?

HINT: It's wrong, but WHY?

134. $g(t) = t^{-1} + 1$ cts on domain.

$t^{-1} = \frac{1}{t}$ so $t \neq 0$ (or we'd have division by zero!)

$\therefore$ g is cts on $(-\infty, 0) \cup (0, \infty)$. ☑

135. $f(x) = \dfrac{5}{e^x - 2}$ cts on domain.

$e^x - 2 = 0 \iff e^x = 2 \iff x = \ln(2)$ ← would be division by 0.

$\therefore$ f is cts on $(-\infty, \ln(2)) \cup (\ln(2), \infty)$ ☑

Note on 135: We haven't talked about e^x or $\ln(x)$ much yet.
    This will become more important later in the course...

136. $f(x) = \dfrac{|x-2|}{x-2}$

Recall $|t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0 \end{cases}$

So $f(x) = \begin{cases} \frac{x-2}{x-2} & \text{if } x-2 \geq 0 \\ \frac{-(x-2)}{x-2} & \text{if } x-2 < 0 \end{cases}$ → if $x-2 = 0$, division by zero!

$= \begin{cases} 1 & \text{if } x-2 > 0 \\ -1 & \text{if } x-2 < 0 \end{cases}$ ←

$= \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$

Now, f is cts on $(-\infty, 2) \cup (2, \infty)$ but NOT at $x=2$
    b/c $f(2)$ is undefined!   ☑

137. $H(x) = \tan(2x)$ cts on domain.

$H(x) = \tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$

So division by zero when...

$\cos(2x) = 0$

$\iff 2x = k\pi + \dfrac{\pi}{2}$ for any integer k.

$\iff x = \left(\dfrac{k}{2} + \dfrac{1}{4}\right)\pi$ for any integer k.

$\therefore$ $H(x)$ is cts when $x \neq \left(\dfrac{k}{2} + \dfrac{1}{4}\right)\pi$ for all integers k. ☑

138. $f(t) = \dfrac{t+3}{t^2+5t+6}$ cts on domain.

$f(t) = \dfrac{t+3}{t^2+5t+6} = \dfrac{t+3}{(t+2)(t+3)}$

So $(t+2)(t+3) = 0$ would give division by zero.

$(t+2)(t+3) = 0 \iff t+2=0$ or $t+3=0$

$\iff t=-2$ or $t=-3$

$\therefore$ f is cts on $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$ ☑ .

Follow-up to 138: Would it have been correct if I

wrote   "$(-\infty, -2) \cup (-3, \infty)$" ?

What about "$(-\infty, -3) \cup (-2, \infty)$" ?

139-144: Is the function continuous at the given point?

139. $f(x) = \dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$.

Not cts at $x = 1$ b/c $f(1)$ is undefined. ☑

140. $h(\theta) = \dfrac{\sin(\theta) - \cos(\theta)}{\tan(\theta)}$ at $\theta = \pi$.

Note $h(\theta) = \dfrac{\sin(\theta) - \cos(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}}$. But $\sin(\pi) = 0$,

So $h(\theta)$ has division by zero at π.

∴ $h(\pi)$ is undefined and h is NOT CTS at $\theta = \pi$. ☑

141.
$$g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1} & \text{if } u \neq \frac{1}{2} \\ \dfrac{7}{2} & \text{if } u = \frac{1}{2} \end{cases} \qquad \text{at } u = \frac{1}{2}.$$

Note $g\left(\frac{1}{2}\right) = \frac{7}{2}$.

Now $\displaystyle\lim_{u \to \frac{1}{2}} g(u) = \lim_{u \to \frac{1}{2}} \frac{6u^2 + u - 2}{2u - 1} \quad \leadsto \quad \frac{0}{0} - \text{type} \text{"}$

$\qquad = \displaystyle\lim_{u \to \frac{1}{2}} \frac{(2u - 1)(3u + 2)}{2u - 1} \quad \leftarrow \text{factored.}$

$\qquad = \displaystyle\lim_{u \to \frac{1}{2}} (3u + 2) = 3 \cdot \frac{1}{2} + 2 = \frac{7}{2}$.

As $\displaystyle\lim_{u \to \frac{1}{2}} g(u) = \frac{7}{2} = g\left(\frac{1}{2}\right)$, we see g is CTS at $u = \frac{1}{2}$ ☑

142. $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$ at $y = 1$.

$\tan(\pi \cdot 1) = \tan(\pi) = 0$, so $f(1)$ undefined.

∴ f is not cts at $y = 1$. ☒

Follow-up to 142: what about at $y = 0$? $y = \frac{1}{2}$?

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$ at $x = 0$.

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x^2 - e^x) = 0^2 - e^0 = -1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x - 1) = 0 - 1 = -1$

$f(0) = 0 - 1 = -1$

As $\lim\limits_{x \to 0^-} f(x) = f(0) = \lim\limits_{x \to 0^+} f(x)$, we have f CTS at 0 ☑

144. $f(x) = \begin{cases} x\sin(x) & \text{if } x \leq \pi \\ x\tan(x) & \text{if } x > \pi \end{cases}$ at $x = \pi$

$\lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} (x\sin(x)) = \pi\sin(\pi) = \pi \cdot 0 = 0$

$\lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} (x\tan(x)) = \pi\tan(\pi) = \pi \cdot 0 = 0$

$f(\pi) = \pi\sin(\pi) = 0$.

As $\lim\limits_{x \to \pi^-} f(x) = f(\pi) = \lim\limits_{x \to \pi^+} f(x)$, we have f cts at $x = \pi$ ☑

145 - 149: Find all values of k for which the function f is cts on its domain.

145. $f(x) = \begin{cases} 3x+2 & \text{if } x < k \\ 2x-3 & \text{if } k \le x \le 8 \end{cases}$

Need $\lim\limits_{x \to k} f(x)$ to exist. So we need

$\lim\limits_{x \to k^-} f(x) = \lim\limits_{x \to k^-} (3x+2) = 3k+2$ and

$\lim\limits_{x \to k^+} f(x) = \lim\limits_{x \to k^+} (2x-3) = 2k-3$ to agree.

i.e. Need $3k+2 = 2k-3$ i.e. $k = -5$.

Left to you: Check that

$f(x) = \begin{cases} 3x+2 & \text{if } x < -5 \\ 2x-3 & \text{if } -5 \le x \le 8 \end{cases}$

is cts at $x = -5$.   ☑

146. $f(\theta) = \begin{cases} \sin(\theta) & 0 \le \theta < \frac{\pi}{2} \\ \cos(\theta+k) & \frac{\pi}{2} \le \theta \le \pi \end{cases}$

Need $\lim\limits_{\theta \to \frac{\pi}{2}} f(\theta)$ to exist. So $\lim\limits_{\theta \to \frac{\pi}{2}^-} \sin(\theta) = \lim\limits_{\theta \to \frac{\pi}{2}^+} \cos(\theta+k)$.

∴ Need $1 = \lim\limits_{\theta \to \frac{\pi}{2}^-} \sin(\theta) = \lim\limits_{\theta \to \frac{\pi}{2}^+} \cos(\theta+k) = \cos\left(\frac{\pi}{2}+k\right)$.

But $\cos(x) = 1$ whenever $x = 2\pi n$ for an integer.

So $\frac{\pi}{2}+k = 2\pi n$ i.e. $k = 2\pi n - \frac{\pi}{2} = \left(\frac{4n-1}{2}\right)\pi$

for any integer n.   ☑

147. $f(x) = \begin{cases} \frac{x^2+3x+2}{x+2} & \text{if } x \neq -2 \\ k & \text{if } x = -2 \end{cases}$

Note $\frac{x^2+3x+2}{x+2} = \frac{(x+1)(x+2)}{x+2} = x+1$ if $x \neq -2$.

So $f(-2) = k$ and $\lim\limits_{x \to -2} f(x) = \lim\limits_{x \to -2} (x+1) = -1$

must agree. Thus $k = -1$. ☑

148. $f(x) = \begin{cases} e^{kx} & \text{if } 0 \leq x < 4 \\ x+3 & \text{if } 4 \leq x \leq 8 \end{cases}$

$\lim\limits_{x \to 4} f(x)$ must exist, so

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} e^{kx} = e^{4k}$ and

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} (x+3) = 7$ must agree.

$\therefore e^{4k} = 7 \iff 4k = \ln(7) \iff k = \frac{\ln(7)}{4}$. ☑

149. $f(x) = \begin{cases} \sqrt{kx} & \text{if } 0 \leq x \leq 3 \\ x+1 & \text{if } 3 < x \leq 10 \end{cases}$

$\lim\limits_{x \to 3} f(x)$ must exist, so

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \sqrt{kx} = \sqrt{3k}$ and

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} x+1 = 4$ must agree.

i.e. $\sqrt{3k} = 4 \iff 3k = 16 \iff k = \frac{16}{3}$. ☑