CONTINUITY

131–138: Find all points of discontinuity.

131: $f(x) = \frac{1}{\sqrt{x}}$ ⟵ Continuous on its domain.

$\sqrt{x} = 0 \iff x = 0^2 = 0$ ⟵ division by 0.

$x \geq 0$ ⟵ Square root of negative.

$\therefore f$ is cts on $(0, \infty)$ $\boxtimes$

132. $f(x) = \frac{2}{x^2+1}$ ⟵ cts on domain

$x^2+1 = 0 \iff x^2 = -1$ ⟵ division by zero.

Reject!

$\therefore f$ is cts on $(-\infty, \infty)$. $\boxtimes$

133. $f(x) = \frac{x}{x^2-x}$ cts on domain.

$x^2 - x = 0 \iff x(x-1) = 0$

$\iff x = 0$ or $x - 1 = 0$

$\iff x = 0$ or $x = 1$ ⟵ Division by zero.

$\therefore f$ is cts on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ $\boxtimes$

Follow-up to 133: Why didn't I write this:

$f(x) = \frac{x}{x^2-x} = \frac{\cancel{x}}{\cancel{x}(x-1)} = \frac{1}{x-1}$, so cts on $(-\infty, 1) \cup (1, \infty)$?

HINT: It's wrong, but WHY?

134. $g(t) = t^{-1} + 1$ cts on domain.

$t^{-1} = \frac{1}{t}$ so $t \neq 0$ (or we'd have division by zero!)

$\therefore$ g is cts on $(-\infty, 0) \cup (0, \infty)$. $\boxtimes$

135. $f(x) = \frac{5}{e^x - 2}$ cts on domain.

$e^x - 2 = 0 \iff e^x = 2 \iff x = \ln(2)$ ← would be division by 0.

$\therefore$ f is cts on $(-\infty, \ln(2)) \cup (\ln(2), \infty)$ $\boxtimes$

Note on 135: We haven't talked about e^x or $\ln(x)$ much yet.

This will become more important later in the course...

136. $f(x) = \frac{|x-2|}{x-2}$

Recall $|t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0 \end{cases}$

So $f(x) = \begin{cases} \frac{x-2}{x-2} & \text{if } x-2 \geq 0 \\ \frac{-(x-2)}{x-2} & \text{if } x-2 < 0 \end{cases}$

if $x-2=0$, division by zero!

$= \begin{cases} 1 & \text{if } x-2 > 0 \\ -1 & \text{if } x-2 < 0 \end{cases}$

$= \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$

Now, f is cts on $(-\infty, 2) \cup (2, \infty)$ but NOT at $x=2$

b/c $f(2)$ is undefined! $\boxtimes$

137. $H(x) = \tan(2x)$ cts on domain.

$H(x) = \tan(2x) = \frac{\sin(2x)}{\cos(2x)}$

So division by zero when...

$\cos(2x) = 0$

$\iff 2x = k\pi + \frac{\pi}{2}$ for any integer k.

$\iff x = \left(\frac{k}{2} + \frac{1}{4}\right)\pi$ for any integer k.

$\therefore H(x)$ is cts when $x \neq \left(\frac{k}{2} + \frac{1}{4}\right)\pi$ for all intgers k. $\boxed{\checkmark}$

138. $f(t) = \frac{t+3}{t^2+5t+6}$ cts on domain.

$f(t) = \frac{t+3}{t^2+5t+6} = \frac{t+3}{(t+2)(t+3)}$

So $(t+2)(t+3) = 0$ would give division by zero.

$(t+2)(t+3) = 0 \iff t+2=0$ or $t+3=0$

$\iff t=-2$ or $t=-3$

$\therefore f$ is cts on $(-\infty,-3) \cup (-3,-2) \cup (-2,\infty)$ $\boxed{\checkmark}$.

Follow-up to 138: Would it have been correct if I wrote "$(-\infty,-2) \cup (-3,\infty)$"?

What about "$(-\infty,-3) \cup (-2,\infty)$"?

139 – 144: Is the function continuous at the given point?

139. $f(x) = \dfrac{2x^2 - 5x + 3}{x-1}$ at $x = 1$.

Not cts at $x=1$ b/c $f(1)$ is undefined. $\boxtimes$

140. $h(\theta) = \dfrac{\sin(\theta) - \cos(\theta)}{\tan(\theta)}$ at $\theta = \pi$.

Note $h(\theta) = \dfrac{\sin(\theta) - \cos(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}}$. But $\sin(\pi) = 0$,

so $h(\theta)$ has division by zero at π.

$\therefore$ $h(\pi)$ is undefined and h is NOT CTS at $\theta = \pi$. $\boxtimes$

141. $g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u-1} & \text{if } u \neq \frac{1}{2} \\ \dfrac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}$ at $u = \frac{1}{2}$.

Note $g(\frac{1}{2}) = \frac{7}{2}$.

Now $\lim\limits_{u \to \frac{1}{2}} g(u) = \lim\limits_{u \to \frac{1}{2}} \dfrac{6u^2 + u - 2}{2u-1} \rightsquigarrow \dfrac{0}{0}$-type !!

$= \lim\limits_{u \to \frac{1}{2}} \dfrac{(2u-1)(3u+2)}{2u-1}$ ← factored.

$= \lim\limits_{u \to \frac{1}{2}} (3u+2) = 3 \cdot \frac{1}{2} + 2 = \frac{7}{2}$.

As $\lim\limits_{u \to \frac{1}{2}} g(u) = \frac{7}{2} = g(\frac{1}{2})$, we see g is CTS at $u = \frac{1}{2}$ $\boxtimes$

142. $f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$ at $y=1$.

$\tan(\pi \cdot 1) = \tan(\pi) = 0$, so $f(1)$ undefined.

$\therefore f$ is not cts at $y=1$. $\boxtimes$

Follow-up to 142: what about at $y=0$? $y=\frac{1}{2}$?

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x<0 \\ x-1 & \text{if } x \geq 0 \end{cases}$ at $x=0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 - e^x) = 0^2 - e^0 = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-1) = 0-1 = -1$$

$$f(0) = 0-1 = -1$$

As $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$, we have f CTS at 0 $\boxtimes$

144. $f(x) = \begin{cases} x\sin(x) & \text{if } x \leq \pi \\ x\tan(x) & \text{if } x > \pi \end{cases}$ at $x=\pi$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^-} (x\sin(x)) = \pi \sin(\pi) = \pi \cdot 0 = 0$$

$$\lim_{x\to \pi^+} f(x) = \lim_{x\to \pi^+} (x\tan(x)) = \pi \tan(\pi) = \pi \cdot 0 = 0$$

$$f(\pi) = \pi \sin(\pi) = 0.$$

As $\lim_{x\to \pi^-} f(x) = f(\pi) = \lim_{x\to \pi^+} f(x)$, we have f cts at $x=\pi$ $\boxtimes$

145 – 149: Find all values of k for which the function f is cts on its domain.

145. $f(x) = \begin{cases} 3x+2 & \text{if } x<k \\ 2x-3 & \text{if } k \le x \le 8 \end{cases}$

Need $\lim_{x\to k} f(x)$ to exist. So we need

$\lim_{x\to k^-} f(x) = \lim_{x\to k^-}(3x+2) = 3k+2$ and

$\lim_{x\to k^+} f(x) = \lim_{x\to k^+}(2x-3) = 2k-3$ to agree.

i.e. Need $3k+2 = 2k-3$ i.e. $k=-5$.

Left to you: Check that

$$f(x) = \begin{cases} 3x+2 & \text{if } x<-5 \\ 2x-3 & \text{if } -5 \le x \le 8 \end{cases}$$

is cts at $x=-5$. $\boxtimes$

146. $f(\theta) = \begin{cases} \sin(\theta) & 0 \le \theta < \frac{\pi}{2} \\ \cos(\theta+k) & \frac{\pi}{2} \le \theta \le \pi \end{cases}$

Need $\lim_{\theta\to\frac{\pi}{2}} f(\theta)$ to exist. So $\lim_{\theta\to\frac{\pi}{2}^-} \sin(\theta) = \lim_{\theta\to\frac{\pi}{2}^+} \cos(\theta+k)$.

$\therefore$ Need $1 = \lim_{\theta\to\frac{\pi}{2}^-} \sin(\theta) = \lim_{\theta\to\frac{\pi}{2}^+} \cos(\theta+k) = \cos\left(\frac{\pi}{2}+k\right)$.

But $\cos(x) = 1$ whenever $x = 2\pi n$ for an integer.

So $\frac{\pi}{2}+k = 2\pi n$ i.e. $k = 2\pi n - \frac{\pi}{2} = \left(\frac{4n-1}{2}\right)\pi$

for any integer n. $\boxtimes$

147. $f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2} & \text{if } x \neq -2 \\ k & \text{if } x = -2 \end{cases}$

Note $\dfrac{x^2+3x+2}{x+2} = \dfrac{(x+1)(x+2)}{x+2} = x+1$ if $x \neq -2$.

So $f(-2) = k$ and $\lim_{x\to -2} f(x) = \lim_{x \to -2}(x+1) = -1$

must agree. Thus $k = -1$. $\boxtimes$

148. $f(x) = \begin{cases} e^{kx} & \text{if } 0 \le x < 4 \\ x+3 & \text{if } 4 \le x \le 8 \end{cases}$

$\lim_{x\to 4} f(x)$ must exist, so

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} e^{kx} = e^{4k}$ and

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x+3) = 7$ must agree.

$\therefore\ e^{4k} = 7 \iff 4k = \ln(7) \iff k = \dfrac{\ln(7)}{4}$. $\boxtimes$

149. $f(x) = \begin{cases} \sqrt{kx} & \text{if } 0 \le x \le 3 \\ x+1 & \text{if } 3 < x \le 10 \end{cases}$

$\lim_{x\to 3} f(x)$ must exist, so

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{kx} = \sqrt{3k}$ and

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+1 = 4$ must agree.

i.e. $\sqrt{3k} = 4 \iff 3k = 16 \iff k = \frac{16}{3}$. $\boxtimes$