

LIMIT CALCULATIONS

83-86: Evaluate via the limit laws.

83. $\lim_{x \rightarrow 0} (4x^2 - 2x + 3)$

$$= \lim_{x \rightarrow 0} (4x^2) - \lim_{x \rightarrow 0} (2x) + \lim_{x \rightarrow 0} 3$$

$$= 4 \lim_{x \rightarrow 0} x^2 - 2 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 3$$

$$= 4 \left(\lim_{x \rightarrow 0} x \right)^2 - 2 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 3$$

$$= 4 \cdot 0^2 - 2 \cdot 0 + \lim_{x \rightarrow 0} 3$$

$$= 4 \cdot 0^2 - 2 \cdot 0 + 3$$

$$= 3$$

LIMIT LAW USED


SUM/DIFFERENCE

CONSTANT MULTIPLE

POWER

LIMIT OF x

CONSTANT

ARITHMETIC 

84. $\lim_{x \rightarrow 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

$$= \frac{\lim_{x \rightarrow 1} (x^3 + 3x^2 + 5)}{\lim_{x \rightarrow 1} (4 - 7x)}$$

$$\lim_{x \rightarrow 1} (4 - 7x)$$

$$= \frac{\lim_{x \rightarrow 1} x^3 + \lim_{x \rightarrow 1} (3x^2) + \lim_{x \rightarrow 1} 5}{\lim_{x \rightarrow 1} 4 - \lim_{x \rightarrow 1} (7x)}$$

$$\lim_{x \rightarrow 1} 4 - \lim_{x \rightarrow 1} (7x)$$

$$= \frac{\lim_{x \rightarrow 1} x^3 + 3 \lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} 5}{\lim_{x \rightarrow 1} 4 - 7 \lim_{x \rightarrow 1} x}$$

$$\lim_{x \rightarrow 1} 4 - 7 \lim_{x \rightarrow 1} x$$

$$= \frac{\left(\lim_{x \rightarrow 1} x \right)^3 + 3 \left(\lim_{x \rightarrow 1} x \right)^2 + \lim_{x \rightarrow 1} 5}{\lim_{x \rightarrow 1} 4 - 7 \lim_{x \rightarrow 1} x}$$

$$\lim_{x \rightarrow 1} 4 - 7 \lim_{x \rightarrow 1} x$$

$$= \frac{1^3 + 3 \cdot 1^2 + 5}{4 - 7 \cdot 1}$$

$$= \frac{9}{-3} = -3$$

LIMIT LAW USED

QUOTIENT

SUM/DIFFERENCE

CONSTANT MULTIPLE

POWER

LIMIT OF x / CONSTANT



$$\begin{aligned}
85. \quad & \lim_{x \rightarrow -2} \sqrt{x^2 - 6x + 3} \\
&= \sqrt{\lim_{x \rightarrow -2} (x^2 - 6x + 3)} \\
&= \sqrt{\lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} (6x) + \lim_{x \rightarrow -2} 3} \\
&= \sqrt{\lim_{x \rightarrow -2} x^2 - 6 \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 3} \\
&= \sqrt{\left(\lim_{x \rightarrow -2} x\right)^2 - 6 \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 3} \\
&= \sqrt{(-2)^2 - 6(-2) + 3} \\
&= \sqrt{4 + 12 + 3} = \sqrt{19}
\end{aligned}$$

LIMIT LAW USED

ROOT

SUM/DIFFERENCE

CONSTANT MULTIPLE

POWER

LIMIT OF x
CONSTANT

ARITHMETIC 

$$\begin{aligned}
86. \quad & \lim_{x \rightarrow -1} (9x + 1)^2 \\
&= \left(\lim_{x \rightarrow -1} (9x + 1)\right)^2 \\
&= \left(\lim_{x \rightarrow -1} (9x) + \lim_{x \rightarrow -1} 1\right)^2 \\
&= \left(9 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 1\right)^2 \\
&= (9(-1) + 1)^2 \\
&= (-8)^2 = 64
\end{aligned}$$

LIMIT LAW USED

POWER

SUM

CONSTANT MULTIPLE

LIMIT OF x
CONSTANT

ARITHMETIC 

87-92: Apply direct substitution to calculate.

NOTE: "direct substitution" works when a function is continuous.

So this works for functions built up from polynomials, roots, quotients, and trigonometric functions on their domains!

$$\underline{87.} \quad \lim_{x \rightarrow 7} x^2 = 7^2 = 49 \quad \square$$

$$\underline{88.} \quad \lim_{x \rightarrow -2} (4x^2 - 1) = 4(-2)^2 - 1 = 15 \quad \square$$

$$\underline{89.} \quad \lim_{x \rightarrow 0} \frac{1}{1 + \sin(x)} = \frac{1}{1 + \sin(0)} = \frac{1}{1+0} = 1 \quad \square$$

$$\underline{90.} \quad \lim_{x \rightarrow 2} e^{2x - x^2} = e^{2 \cdot 2 - 2^2} = e^0 = 1 \quad \square$$

$$\underline{91.} \quad \lim_{x \rightarrow 1} \frac{2 - 7x}{x + 6} = \frac{2 - 7 \cdot 1}{1 + 6} = -\frac{5}{7} \quad \square$$

$$\underline{92.} \quad \lim_{x \rightarrow 3} \ln(e^{3x}) = \ln(e^{3 \cdot 3}) = \ln(e^9) = 9 \quad \square$$

93-102: Show the limit has " $\frac{0}{0}$ " type, then evaluate.

$$\underline{93.} \quad \lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} \rightsquigarrow \frac{4^2 - 16}{4 - 4} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{x \rightarrow 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x \rightarrow 4} (x+4) = 4+4 = 8 \quad \square$$

$$\underline{94.} \quad \lim_{x \rightarrow 2} \frac{x-2}{x^2-2x} \rightsquigarrow \frac{2-2}{2^2-2 \cdot 2} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{x \rightarrow 2} \frac{x-2}{x(x-2)} = \lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2} \quad \square$$

$$\underline{95.} \quad \lim_{x \rightarrow 6} \frac{3x-18}{2x-12} \rightsquigarrow \frac{3 \cdot 6 - 18}{2 \cdot 6 - 12} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{x \rightarrow 6} \frac{3(x-6)}{2(x-6)} = \lim_{x \rightarrow 6} \frac{3}{2} = \frac{3}{2} \quad \square$$

$$\underline{96.} \quad \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h} \rightsquigarrow \frac{(1+0)^2 - 1}{0} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{h \rightarrow 0} \frac{1+2h+h^2-1}{h} = \lim_{h \rightarrow 0} \frac{h(2+h)}{h} = \lim_{h \rightarrow 0} (2+h) = 2+0 = 2 \quad \square$$

$$\underline{97.} \quad \lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} \rightsquigarrow \frac{9-9}{\sqrt{9}-3} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{t \rightarrow 9} \frac{(\sqrt{t})^2 - 3^2}{\sqrt{t}-3} = \lim_{t \rightarrow 9} \frac{(\sqrt{t}+3)(\sqrt{t}-3)}{\sqrt{t}-3} = \lim_{t \rightarrow 9} (\sqrt{t}+3) = \sqrt{9}+3 = 6 \quad \square$$

$$\underline{97 (ALT)}. \quad \lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} \rightsquigarrow \frac{9-9}{\sqrt{9}-3} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$\lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} = \lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} \cdot \frac{\sqrt{t}+3}{\sqrt{t}+3} = \lim_{t \rightarrow 9} \frac{(t-9)(\sqrt{t}+3)}{(\sqrt{t})^2 - 3^2}$$

$$= \lim_{t \rightarrow 9} \frac{(t-9)(\sqrt{t}+3)}{t-9} = \lim_{t \rightarrow 9} (\sqrt{t}+3) = \sqrt{9}+3 = 6 \quad \square$$

98. Let a be a nonzero, constant real number.

$$\lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \rightsquigarrow \frac{\frac{1}{a+0} - \frac{1}{a}}{0} \rightsquigarrow \frac{0}{0} \text{-type!}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} \cdot \frac{a}{a} - \frac{1}{a} \cdot \frac{a+h}{a+h}}{h} = \lim_{h \rightarrow 0} \frac{\frac{a - (a+h)}{a(a+h)}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a - a - h}{a(a+h)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-h}{ha(a+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} = \frac{-1}{a(a+0)} = \frac{-1}{a^2} \quad \square$$

$$\underline{99.} \quad \lim_{\theta \rightarrow \pi} \frac{\sin(\theta)}{\tan(\theta)} \rightsquigarrow \frac{0}{0} - \text{type!}$$

$$= \lim_{\theta \rightarrow \pi} \frac{\sin(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}} = \lim_{\theta \rightarrow \pi} \left(\sin(\theta) \cdot \frac{\cos(\theta)}{\sin(\theta)} \right) = \lim_{\theta \rightarrow \pi} \cos(\theta) = \cos(\pi) = -1 \quad \square$$

$$\underline{100.} \quad \lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} \rightsquigarrow \frac{1^3 - 1}{1^2 - 1} \rightsquigarrow \frac{0}{0} - \text{type!}$$

$$\text{Note: } x^3 - 1 = x^3 - 1^3 = (x-1)(x^2 + x \cdot 1 + 1^2) = (x-1)(x^2 + x + 1)$$

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x+1} = \frac{1^2 + 1 + 1}{1+1} = \frac{3}{2} \quad \square$$

$$\underline{101.} \quad \lim_{x \rightarrow \frac{1}{2}} \frac{2x^2 + 3x - 2}{2x - 1} \rightsquigarrow \frac{2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2}{2\left(\frac{1}{2}\right) - 1} \rightsquigarrow \frac{2 \cdot \frac{1}{4} + \frac{3}{2} - 2}{1 - 1} \rightsquigarrow \frac{0}{0} - \text{type!}$$

$$\text{Factor: } 2x^2 + 3x - 2 = (2x - 1)(x + 2) \leftarrow \text{check!}$$

$$\lim_{x \rightarrow \frac{1}{2}} \frac{2x^2 + 3x - 2}{2x - 1} = \lim_{x \rightarrow \frac{1}{2}} \frac{(2x - 1)(x + 2)}{2x - 1} = \lim_{x \rightarrow \frac{1}{2}} (x + 2) = \frac{1}{2} + 2 = \frac{5}{2} \quad \square$$

$$\underline{102.} \quad \lim_{x \rightarrow -3} \frac{\sqrt{x+4} - 1}{x+3} \rightsquigarrow \frac{\sqrt{-3+4} - 1}{-3+3} \rightsquigarrow \frac{\sqrt{1} - 1}{0} \rightsquigarrow \frac{0}{0} - \text{type!}$$

$$= \lim_{x \rightarrow -3} \left(\frac{\sqrt{x+4} - 1}{x+3} \cdot \frac{\sqrt{x+4} + 1}{\sqrt{x+4} + 1} \right)$$

$$= \lim_{x \rightarrow -3} \frac{(x+4) - 1}{(x+3)(\sqrt{x+4} + 1)}$$

$$= \lim_{x \rightarrow -3} \frac{x+3}{(x+3)(\sqrt{x+4} + 1)}$$

$$= \lim_{x \rightarrow -3} \frac{1}{\sqrt{x+4} + 1} = \frac{1}{\sqrt{-3+4} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} \quad \square$$

103-106: Calculate the limit. If the limit DNE, give a complete explanation using methods from class.
Be as specific as possible.

$$\underline{103.} \quad \lim_{x \rightarrow -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2} \xrightarrow[\text{SUBSTITUTION}]{\text{TRY}} \frac{2(-2)^2 + 7(-2) - 4}{(-2)^2 + (-2) - 2} \rightsquigarrow \frac{8 - 14 - 4}{4 - 2 - 2} \rightsquigarrow \frac{-10}{0} \rightsquigarrow \infty$$

Rewrite:

$$\begin{aligned} \lim_{x \rightarrow -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2} &= \lim_{x \rightarrow -2^-} \frac{2x^2 + 7x - 4}{(x+2)(x-1)} \\ &= \lim_{x \rightarrow -2^-} \left(\frac{2x^2 + 7x - 4}{x-1} \cdot \frac{1}{x+2} \right) \end{aligned}$$

Note: $\lim_{x \rightarrow -2^-} \frac{2x^2 + 7x - 4}{x-1} = \frac{2(-2)^2 + 7(-2) - 4}{(-2) - 1} = \frac{-10}{-3} = \frac{10}{3} > 0$

and $x+2 \rightarrow 0^-$ as $x \rightarrow -2^-$, so $\lim_{x \rightarrow -2^-} \frac{1}{x+2} = -\infty$.

Thus $\lim_{x \rightarrow -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = -\infty$ \square

104. $\lim_{x \rightarrow -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2} \rightsquigarrow \frac{-10}{0}$ - type

Also, $\lim_{x \rightarrow -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \lim_{x \rightarrow -2^+} \left(\frac{2x^2 + 7x - 4}{x-1} \cdot \frac{1}{x+2} \right)$ } Same as in 103.

and $\lim_{x \rightarrow -2^+} \frac{2x^2 + 7x - 4}{x-1} = \frac{10}{3} > 0$

Now $\lim_{x \rightarrow -2^+} \frac{1}{x+2} = \infty$ b/c $x+2 \rightarrow 0^+$ as $x \rightarrow -2^+$, } Different!

So $\lim_{x \rightarrow -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \infty$ \square

$$105. \lim_{x \rightarrow 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2} \xrightarrow[\text{SUBSTITUTION}]{\text{TRY}} \frac{2 \cdot 1^2 + 7 \cdot 1 - 4}{1^2 + 1 - 2} \rightsquigarrow \frac{2+7-4}{2-2} \rightsquigarrow \frac{5}{0} \text{-type}$$

$$= \lim_{x \rightarrow 1^-} \frac{2x^2 + 7x - 4}{(x+2)(x-1)} = \lim_{x \rightarrow 1^-} \left(\frac{2x^2 + 7x - 4}{x+2} \cdot \frac{1}{x-1} \right)$$

$$\text{Now } \lim_{x \rightarrow 1^-} \frac{2x^2 + 7x - 4}{x+2} = \frac{2 \cdot 1^2 + 7 \cdot 1 - 4}{1+2} = \frac{5}{3} > 0$$

and as $x \rightarrow 1^-$ we have $x-1 \rightarrow 0^-$, so $\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$.

By a proposition from class, $\lim_{x \rightarrow 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = -\infty$. \square

$$106. \lim_{x \rightarrow 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2} \rightsquigarrow \frac{5}{0} \text{-type}$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2x^2 + 7x - 4}{x+2} \cdot \frac{1}{x-1} \right)$$

Same as 105

$$\lim_{x \rightarrow 1^+} \frac{2x^2 + 7x - 4}{x+2} = \frac{5}{3} > 0$$

As $x \rightarrow 1^+$, $x-1 \rightarrow 0^+$, so $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$.

$$\therefore \lim_{x \rightarrow 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \infty \quad \square$$

The same?
Well, yes. But
actually, no.

107-114: Suppose $\lim_{x \rightarrow 6} f(x) = 4$, $\lim_{x \rightarrow 6} g(x) = 9$, and $\lim_{x \rightarrow 6} h(x) = 6$. Evaluate!

107. $\lim_{x \rightarrow 6} (2f(x)g(x))$

$$= 2 \cdot \lim_{x \rightarrow 6} (f(x) \cdot g(x))$$

$$= 2 \cdot \lim_{x \rightarrow 6} f(x) \cdot \lim_{x \rightarrow 6} g(x)$$

$$= 2 \cdot 4 \cdot 9 = 72$$

LIMIT LAW

CONSTANT MULTIPLE

PRODUCT

ARITHMETIC \square

108. $\lim_{x \rightarrow 6} \frac{g(x) - 1}{f(x)}$

$$= \frac{\lim_{x \rightarrow 6} (g(x) - 1)}{\lim_{x \rightarrow 6} f(x)}$$

$$= \frac{\lim_{x \rightarrow 6} g(x) - \lim_{x \rightarrow 6} 1}{\lim_{x \rightarrow 6} f(x)}$$

$$= \frac{\lim_{x \rightarrow 6} g(x) - 1}{\lim_{x \rightarrow 6} f(x)}$$

$$= \frac{9 - 1}{4} = \frac{8}{4} = 2$$

LIMIT LAW USED

QUOTIENT

DIFFERENCE

CONSTANT

ARITHMETIC \square

109. $\lim_{x \rightarrow 6} (f(x) + \frac{1}{3}g(x))$

$$= \lim_{x \rightarrow 6} f(x) + \lim_{x \rightarrow 6} (\frac{1}{3}g(x))$$

$$= \lim_{x \rightarrow 6} f(x) + \frac{1}{3} \lim_{x \rightarrow 6} g(x)$$

$$= 4 + \frac{1}{3} \cdot 9 = 4 + 3 = 7$$

LIMIT LAW USED

SUM

CONSTANT MULTIPLE

ARITHMETIC

$$\begin{aligned}
 \underline{110.} \quad & \lim_{x \rightarrow 6} \frac{(h(x))^3}{2} \\
 &= \frac{1}{2} \lim_{x \rightarrow 6} (h(x))^3 \\
 &= \frac{1}{2} \left(\lim_{x \rightarrow 6} h(x) \right)^3 \\
 &= \frac{1}{2} \cdot 6^3 = \frac{1}{2} \cdot 216 = 108
 \end{aligned}$$

LIMIT LAW

CONSTANT MULTIPLE
POWER

ARITHMETIC \square

$$\begin{aligned}
 \underline{111.} \quad & \lim_{x \rightarrow 6} \sqrt{g(x) - f(x)} \\
 &= \sqrt{\lim_{x \rightarrow 6} (g(x) - f(x))} \\
 &= \sqrt{\lim_{x \rightarrow 6} g(x) - \lim_{x \rightarrow 6} f(x)} \\
 &= \sqrt{9 - 4} = \sqrt{5}
 \end{aligned}$$

LIMIT LAW

ROOT

DIFFERENCE

ARITHMETIC \square

$$\begin{aligned}
 \underline{112.} \quad & \lim_{x \rightarrow 6} (x \cdot h(x)) \\
 &= \left(\lim_{x \rightarrow 6} x \right) \cdot \left(\lim_{x \rightarrow 6} h(x) \right) \\
 &= 6 \cdot \lim_{x \rightarrow 6} h(x) \\
 &= 6 \cdot 6 = 36
 \end{aligned}$$

LAW

PRODUCT

LIMIT OF x

ARITHMETIC \square

$$\begin{aligned}
 \underline{113.} \quad & \lim_{x \rightarrow 6} ((x+1) \cdot f(x)) \\
 &= \lim_{x \rightarrow 6} (x+1) \cdot \lim_{x \rightarrow 6} f(x) \\
 &= \left(\lim_{x \rightarrow 6} x + \lim_{x \rightarrow 6} 1 \right) \cdot \lim_{x \rightarrow 6} f(x) \\
 &= (6 + 1) \cdot \lim_{x \rightarrow 6} f(x) \\
 &= (6+1) \cdot 4 = 7 \cdot 4 = 28
 \end{aligned}$$

LAW

PRODUCT

SUM

LIMIT OF x
CONSTANT

ARITHMETIC \square

$$\underline{114.} \quad \lim_{x \rightarrow 6} (f(x) \cdot g(x) - h(x))$$

$$= \lim_{x \rightarrow 6} (f(x) \cdot g(x)) - \lim_{x \rightarrow 6} h(x)$$

$$= \lim_{x \rightarrow 6} f(x) \cdot \lim_{x \rightarrow 6} g(x) - \lim_{x \rightarrow 6} h(x)$$

$$= 4 \cdot 9 - 6 = 36 - 6 = 30$$

LAW

DIFFERENCE

PRODUCT

ARITHMETIC \square

$$\underline{127.} \quad \text{Evaluate } \lim_{\theta \rightarrow 0} \theta^2 \cos\left(\frac{1}{\theta}\right).$$

Note $-1 \leq \cos(x) \leq 1$ for all x .

So $-1 \leq \cos\left(\frac{1}{\theta}\right) \leq 1$ for all θ .

But $\theta^2 > 0$ for all $\theta \neq 0$.

So $-\theta^2 \leq \theta^2 \cos\left(\frac{1}{\theta}\right) \leq \theta^2$ for all $\theta \neq 0$

$\therefore \lim_{\theta \rightarrow 0} (-\theta^2) \leq \lim_{\theta \rightarrow 0} \theta^2 \cos\left(\frac{1}{\theta}\right) \leq \lim_{\theta \rightarrow 0} \theta^2$ if the limits exist

$\therefore 0 = -0^2 \leq \lim_{\theta \rightarrow 0} \theta^2 \cos\left(\frac{1}{\theta}\right) \leq 0^2 = 0$.

By the Squeeze Theorem $\lim_{\theta \rightarrow 0} \theta^2 \cos\left(\frac{1}{\theta}\right) = 0$ exists! \square