# LIMIT CALCULATIONS

**83–86:** Evaluate via the limit laws.

**83.** $\lim_{x\to 0}(4x^2-2x+3)$

| | Limit Law Used |
|---|---|
| $=\lim_{x\to 0}(4x^2)-\lim_{x\to 0}(2x)+\lim_{x\to 0}3$ | Sum/Difference |
| $=4\lim_{x\to 0}x^2-2\lim_{x\to 0}x+\lim_{x\to 0}3$ | Constant Multiple |
| $=4\left(\lim_{x\to 0}x\right)^2-2\lim_{x\to 0}x+\lim_{x\to 0}3$ | Power |
| $=4\cdot 0^2-2\cdot 0+\lim_{x\to 0}3$ | Limit of $x$ |
| $=4\cdot 0^2-2\cdot 0+3$ | Constant |
| $=3$ | Arithmetic |

**84.** $\lim_{x\to 1}\dfrac{x^3+3x^2+5}{4-7x}$

| | Limit Law Used |
|---|---|
| $=\dfrac{\lim_{x\to 1}(x^3+3x^2+5)}{\lim_{x\to 1}(4-7x)}$ | Quotient |
| $=\dfrac{\lim_{x\to 1}x^3+\lim_{x\to 1}(3x^2)+\lim_{x\to 1}5}{\lim_{x\to 1}4-\lim_{x\to 1}(7x)}$ | Sum/Difference |
| $=\dfrac{\lim_{x\to 1}x^3+3\lim_{x\to 1}x^2+\lim_{x\to 1}5}{\lim_{x\to 1}4-7\lim_{x\to 1}x}$ | Constant Multiple |
| $=\dfrac{\left(\lim_{x\to 1}x\right)^3+3\left(\lim_{x\to 1}x\right)^2+\lim_{x\to 1}5}{\lim_{x\to 1}4-7\lim_{x\to 1}x}$ | Power |
| $=\dfrac{1^3+3\cdot 1^2+5}{4-7\cdot 1}$ | Limit of $x$/Constant |
| $=\dfrac{9}{-3}=-3$ | |

85. 

| $\lim\limits_{x\to -2}\sqrt{x^2-6x+3}$ | LIMIT LAW USED |
| --- | --- |
| $=\sqrt{\lim\limits_{x\to -2}(x^2-6x+3)}$ | ROOT |
| $=\sqrt{\lim\limits_{x\to -2}x^2-\lim\limits_{x\to -2}(6x)+\lim\limits_{x\to -2}3}$ | SUM/DIFFERENCE |
| $=\sqrt{\lim\limits_{x\to -2}x^2-6\lim\limits_{x\to -2}x+\lim\limits_{x\to -2}3}$ | CONSTANT MULTIPLE |
| $=\sqrt{\left(\lim\limits_{x\to -2}x\right)^2-6\lim\limits_{x\to -2}x+\lim\limits_{x\to -2}3}$ | POWER |
| $=\sqrt{(-2)^2-6(-2)+3}$ | LIMIT OF $x$ <br> CONSTANT |
| $=\sqrt{4+12+3}=\sqrt{19}$ | ARITHMETIC |

86. 

| $\lim\limits_{x\to -1}(9x+1)^2$ | LIMIT LAW USED |
| --- | --- |
| $=\left(\lim\limits_{x\to -1}(9x+1)\right)^2$ | POWER |
| $=\left(\lim\limits_{x\to -1}(9x)+\lim\limits_{x\to -1}1\right)^2$ | SUM |
| $=\left(9\lim\limits_{x\to -1}x+\lim\limits_{x\to -1}1\right)^2$ | CONSTANT MULTIPLE |
| $=(9(-1)+1)^2$ | LIMIT OF $x$ <br> CONSTANT |
| $=(-8)^2=64$ | ARITHMETIC |

**87–92:** Apply direct substitution to calculate.

**NOTE:** "direct substitution" works when a function is continuous. So this works for functions built up from polynomials, roots, quotients, and trigonometric functions on their domains!

**87.** $\lim\limits_{x\to 7} x^2 = 7^2 = 49$ $\blacksquare$

**88.** $\lim\limits_{x\to -2} (4x^2-1) = 4(-2)^2 - 1 = 15$ $\blacksquare$

**89.** $\lim\limits_{x\to 0} \dfrac{1}{1+\sin(x)} = \dfrac{1}{1+\sin(0)} = \dfrac{1}{1+0} = 1$ $\blacksquare$

**90.** $\lim\limits_{x\to 2} e^{2x-x^2} = e^{2\cdot 2 - 2^2} = e^0 = 1$ $\blacksquare$

**91.** $\lim\limits_{x\to 1} \dfrac{2-7x}{x+6} = \dfrac{2-7\cdot 1}{1+6} = -\dfrac{5}{7}$ $\blacksquare$

**92.** $\lim\limits_{x\to 3} \ln(e^{3x}) = \ln(e^{3\cdot 3}) = \ln(e^9) = 9$ $\blacksquare$

**93–102:** Show the limit has "$\frac{0}{0}$"-type, then evaluate.

**93.** $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} \rightsquigarrow \dfrac{4^2-16}{4-4} \rightsquigarrow \dfrac{0}{0}$-type!

$$= \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x\to 4} (x+4) = 4+4 = 8 \quad \blacksquare$$

**94.** $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-2x} \rightsquigarrow \dfrac{2-2}{2^2-2\cdot 2} \rightsquigarrow \dfrac{0}{0}$-type!

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)} = \lim_{x\to 2} \frac{1}{x} = \frac{1}{2} \quad \blacksquare$$

95. $\lim_{x\to 6} \frac{3x-18}{2x-12} \rightsquigarrow \frac{3\cdot 6-18}{2\cdot 6-12} \rightsquigarrow \frac{0}{0}$-type!

$= \lim_{x\to 6} \frac{3(x-6)}{2(x-6)} = \lim_{x\to 6} \frac{3}{2} = \frac{3}{2}$ ☒

96. $\lim_{h\to 0} \frac{(1+h)^2-1}{h} \rightsquigarrow \frac{(1+0)^2-1}{0} \rightsquigarrow \frac{0}{0}$-type!

$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2+0 = 2$ ☒

97. $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} \rightsquigarrow \frac{9-9}{\sqrt{9}-3} \rightsquigarrow \frac{0}{0}$-type!

$= \lim_{t\to 9} \frac{(\sqrt{t})^2-3^2}{\sqrt{t}-3} = \lim_{t\to 9} \frac{(\sqrt{t}+3)(\sqrt{t}-3)}{\sqrt{t}-3} = \lim_{t\to 9}(\sqrt{t}+3) = \sqrt{9}+3 = 6$ ☒

97 (ALT). $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} \rightsquigarrow \frac{9-9}{\sqrt{9}-3} \rightsquigarrow \frac{0}{0}$-type!

$\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = \lim_{t\to 9} \frac{t-9}{\sqrt{t}-3}\cdot\frac{\sqrt{t}+3}{\sqrt{t}+3} = \lim_{t\to 9} \frac{(t-9)(\sqrt{t}+3)}{(\sqrt{t})^2-3^2}$

$= \lim_{t\to 9} \frac{(t-9)(\sqrt{t}+3)}{t-9} = \lim_{t\to 9}(\sqrt{t}+3) = \sqrt{9}+3 = 6$ ☒

98. Let $a$ be a nonzero, constant real number.

$\lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} \rightsquigarrow \frac{\frac{1}{a+0}-\frac{1}{a}}{0} \rightsquigarrow \frac{0}{0}$-type!

$= \lim_{h\to 0} \frac{\frac{1}{a+h}\cdot\frac{a}{a} - \frac{1}{a}\cdot\frac{a+h}{a+h}}{h} = \lim_{h\to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h}$

$= \lim_{h\to 0} \frac{a-a-h}{a(a+h)}\cdot\frac{1}{h} = \lim_{h\to 0} \frac{-h}{ha(a+h)}$

$= \lim_{h\to 0} \frac{-1}{a(a+h)} = \frac{-1}{a(a+0)} = \frac{-1}{a^2}$ ☒

99. $\lim_{\theta \to \pi} \frac{\sin(\theta)}{\tan(\theta)} \rightsquigarrow \frac{0}{0}$ - type!

$$= \lim_{\theta \to \pi} \frac{\sin(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}} = \lim_{\theta \to \pi} \left( \sin(\theta) \cdot \frac{\cos(\theta)}{\sin(\theta)} \right) = \lim_{\theta \to \pi} \cos(\theta) = \cos(\pi) = -1 \quad \boxtimes$$

100. $\lim_{x \to 1} \frac{x^3-1}{x^2-1} \rightsquigarrow \frac{1^3-1}{1^2-1} \rightsquigarrow \frac{0}{0}$ - type!

Note: $x^3 - 1 = x^3 - 1^3 = (x-1)(x^2 + x \cdot 1 + 1^2) = (x-1)(x^2+x+1)$

$$\lim_{x \to 1} \frac{x^3-1}{x^2-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x^2+x+1}{x+1} = \frac{1^2+1+1}{1+1} = \frac{3}{2} \quad \boxtimes$$

101. $\lim_{x \to 1/2} \frac{2x^2+3x-2}{2x-1} \rightsquigarrow \frac{2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2}{2\left(\frac{1}{2}\right)-1} \rightsquigarrow \frac{2 \cdot \frac{1}{4} + \frac{3}{2} - 2}{1-1} \rightsquigarrow \frac{0}{0}$ - type!

Factor: $2x^2+3x-2 = (2x-1)(x+2)$ ← check!

$$\lim_{x \to 1/2} \frac{2x^2+3x-2}{2x-1} = \lim_{x \to \frac{1}{2}} \frac{(2x-1)(x+2)}{2x-1} = \lim_{x \to \frac{1}{2}} (x+2) = \frac{1}{2} + 2 = \frac{5}{2} \quad \boxtimes$$

102. $\lim_{x \to -3} \frac{\sqrt{x+4}-1}{x+3} \rightsquigarrow \frac{\sqrt{-3+4}-1}{-3+3} \rightsquigarrow \frac{\sqrt{1}-1}{0} \rightsquigarrow \frac{0}{0}$ - type!

$$= \lim_{x \to -3} \left( \frac{\sqrt{x+4}-1}{x+3} \cdot \frac{\sqrt{x+4}+1}{\sqrt{x+4}+1} \right)$$

$$= \lim_{x \to -3} \frac{(x+4)-1}{(x+3)(\sqrt{x+4}+1)}$$

$$= \lim_{x \to -3} \frac{x+3}{(x+3)(\sqrt{x+4}+1)}$$

$$= \lim_{x \to -3} \frac{1}{\sqrt{x+4}+1} = \frac{1}{\sqrt{-3+4}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2} \quad \boxtimes$$

**103–106:** Calculate the limit. If the limit DNE, give a complete explanation using methods from class.
Be as specific as possible.

**103.** $\displaystyle \lim_{x\to-2^-} \frac{2x^2+7x-4}{x^2+x-2} \xrightarrow[\text{SUBSTITUTION}]{\text{TRY}} \frac{2(-2)^2+7(-2)-4}{(-2)^2+(-2)-2} \rightsquigarrow \frac{8-14-4}{4-2-2} \rightsquigarrow \frac{-10}{0}$ :(

**Rewrite:**

$$\lim_{x\to-2^-} \frac{2x^2+7x-4}{x^2+x-2}$$

$$= \lim_{x\to-2^-} \frac{2x^2+7x-4}{(x+2)(x-1)}$$

$$= \lim_{x\to-2^-} \left(\frac{2x^2+7x-4}{x-1}\cdot\frac{1}{x+2}\right)$$

**Note:** $\displaystyle \lim_{x\to-2^-} \frac{2x^2+7x-4}{x-1} = \frac{2(-2)^2+7(-2)-4}{(-2)-1} = \frac{-10}{-3} = \frac{10}{3} > 0$

and $x+2 \to 0^-$ as $x\to-2^-$, so $\displaystyle \lim_{x\to-2^-}\frac{1}{x+2} = -\infty$.

Thus $\displaystyle \lim_{x\to-2^-} \frac{2x^2+7x-4}{x^2+x-2} = -\infty$ $\square$

**104.** $\displaystyle \lim_{x\to-2^+} \frac{2x^2+7x-4}{x^2+x-2} \rightsquigarrow \frac{-10}{0}$ - type

Also, $\displaystyle \lim_{x\to-2^+} \frac{2x^2+7x-4}{x^2+x-2} = \lim_{x\to-2^+}\left(\frac{2x^2+7x-4}{x-1}\cdot\frac{1}{x+2}\right)$,

and $\displaystyle \lim_{x\to-2^+} \frac{2x^2+7x-4}{x-1} = \frac{10}{3} > 0$

Same as in 103.

Now $\displaystyle \lim_{x\to-2^+}\frac{1}{x+2} = \infty$ b/c $x+2\to0^+$ as $x\to-2^+$,

So $\displaystyle \lim_{x\to-2^+} \frac{2x^2+7x-4}{x^2+x-2} = \infty$ $\square$

Different!

**105.** $\lim_{x\to1^-} \frac{2x^2+7x-4}{x^2+x-2} \xrightarrow[\text{SUBSTITUTION}]{\text{TRY}} \frac{2\cdot1^2+7\cdot1-4}{1^2+1-2} \rightsquigarrow \frac{2+7-4}{2-2} \rightsquigarrow \frac{5}{0}$-type

$$= \lim_{x\to1^-} \frac{2x^2+7x-4}{(x+2)(x-1)} = \lim_{x\to1^-}\left(\frac{2x^2+7x-4}{x+2}\cdot\frac{1}{x-1}\right)$$

Now $\lim_{x\to1^-} \frac{2x^2+7x-4}{x+2} = \frac{2\cdot1^2+7\cdot1-4}{1+2} = \frac{5}{3} > 0$

and as $x\to1^-$ we have $x-1\to0^-$, so $\lim_{x\to1^-}\frac{1}{x-1} = -\infty$.

By a proposition from class, $\lim_{x\to1^-}\frac{2x^2+7x-4}{x^2+x-2} = -\infty$. $\boxtimes$

**106.** $\lim_{x\to1^+} \frac{2x^2+7x-4}{x^2+x-2} \rightsquigarrow \frac{5}{0}$-type...

$$= \lim_{x\to1^+}\left(\frac{2x^2+7x-4}{x+2}\cdot\frac{1}{x-1}\right)$$

$$\lim_{x\to1^+}\frac{2x^2+7x-4}{x+2} = \frac{5}{3} > 0$$

Same as 105

As $x\to1^+$, $x-1\to0^+$, so $\lim_{x\to1^+}\frac{1}{x-1} = \infty$.

$\therefore \lim_{x\to1^+}\frac{2x^2+7x-4}{x^2+x-2} = \infty$ $\boxtimes$

The same? Well, yes. But actually, no.

107–114: Suppose $\lim_{x\to 6} f(x) = 4$, $\lim_{x\to 6} g(x) = 9$, and $\lim_{x\to 6} h(x) = 6$. Evaluate!

107. $\lim_{x\to 6}\left(2f(x)g(x)\right)$

| | LIMIT LAW |
|---|---|
| $= 2\cdot\lim_{x\to 6}\left(f(x)\cdot g(x)\right)$ | CONSTANT MULTIPLE |
| $= 2\cdot\lim_{x\to 6} f(x)\cdot\lim_{x\to 6} g(x)$ | PRODUCT |
| $= 2\cdot 4\cdot 9 = 72$ | ARITHMETIC $\boxtimes$ |

108. $\lim_{x\to 6}\dfrac{g(x)-1}{f(x)}$

| | LIMIT LAW USED |
|---|---|
| $= \dfrac{\lim_{x\to 6}\left(g(x)-1\right)}{\lim_{x\to 6} f(x)}$ | QUOTIENT |
| $= \dfrac{\lim_{x\to 6} g(x) - \lim_{x\to 6} 1}{\lim_{x\to 6} f(x)}$ | DIFFERENCE |
| $= \dfrac{\lim_{x\to 6} g(x) - 1}{\lim_{x\to 6} f(x)}$ | CONSTANT |
| $= \dfrac{9-1}{4} = \dfrac{8}{4} = 2$ | ARITHMETIC $\boxtimes$ |

109. $\lim_{x\to 6}\left(f(x) + \frac{1}{3}g(x)\right)$

| | LIMIT LAW USED |
|---|---|
| $= \lim_{x\to 6} f(x) + \lim_{x\to 6}\left(\frac{1}{3}g(x)\right)$ | SUM |
| $= \lim_{x\to 6} f(x) + \frac{1}{3}\lim_{x\to 6} g(x)$ | CONSTANT MULTIPLE |
| $= 4 + \frac{1}{3}\cdot 9 = 4+3 = 7$ | ARITHMETIC |

| 110. $\lim_{x\to 6} \frac{(h(x))^3}{2}$ | Limit Law |
|---|---|
| $= \frac{1}{2} \lim_{x\to 6} (h(x))^3$ | Constant Multiple |
| $= \frac{1}{2} \left( \lim_{x\to 6} h(x) \right)^3$ | Power |
| $= \frac{1}{2} \cdot 6^3 = \frac{1}{2} \cdot 216 = 108$ | Arithmetic |

| 111. $\lim_{x\to 6} \sqrt{g(x) - f(x)}$ | Limit Law |
|---|---|
| $= \sqrt{\lim_{x\to 6} \left( g(x) - f(x) \right)}$ | Root |
| $= \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)}$ | Difference |
| $= \sqrt{9 - 4} = \sqrt{5}$ | Arithmetic |

| 112. $\lim_{x\to 6} \left( x \cdot h(x) \right)$ | Law |
|---|---|
| $= \left( \lim_{x\to 6} x \right) \cdot \left( \lim_{x\to 6} h(x) \right)$ | Product |
| $= 6 \cdot \lim_{x\to 6} h(x)$ | Limit of $x$ |
| $= 6 \cdot 6 = 36$ | Arithmetic |

| 113. $\lim_{x\to 6} \left( (x+1) \cdot f(x) \right)$ | Law |
|---|---|
| $= \lim_{x\to 6} (x+1) \cdot \lim_{x\to 6} f(x)$ | Product |
| $= \left( \lim_{x\to 6} x + \lim_{x\to 6} 1 \right) \cdot \lim_{x\to 6} f(x)$ | Sum |
| $= (6 + 1) \cdot \lim_{x\to 6} f(x)$ | Limit of $x$<br>Constant |
| $= (6+1) \cdot 4 = 7 \cdot 4 = 28$ | Arithmetic |

114. $\lim\limits_{x\to 6}\left(f(x)\cdot g(x) - h(x)\right)$ LAW

$= \lim\limits_{x\to 6}\left(f(x)\cdot g(x)\right) - \lim\limits_{x\to 6} h(x)$ DIFFERENCE

$= \lim\limits_{x\to 6} f(x) \cdot \lim\limits_{x\to 6} g(x) - \lim\limits_{x\to 6} h(x)$ PRODUCT

$= 4\cdot 9 - 6 = 36 - 6 = 30$ ARITHMETIC

127. Evaluate $\lim\limits_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$.

Note $-1 \le \cos(x) \le 1$ for all $x$.

So $-1 \le \cos\left(\frac{1}{\theta}\right) \le 1$ for all $\theta$.

But $\theta^2 > 0$ for all $\theta \neq 0$.

So $-\theta^2 \le \theta^2 \cos\left(\frac{1}{\theta}\right) \le \theta^2$ for all $\theta \neq 0$

$\therefore \lim\limits_{\theta\to 0}(-\theta^2) \le \lim\limits_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right) \le \lim\limits_{\theta\to 0} \theta^2$ if the limits exist

$\therefore 0 = -0^2 \le \lim\limits_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right) \le 0^2 = 0$.

By the Squeeze Theorem $\lim\limits_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right) = 0$ exists!