COMPUTING DERIVATIVES

Recall: for function f defined on an interval containing a, The derivative of f at a is (provided it exists):

$$f'(a) = \lim_{x \to a} \underbrace{\frac{f(x) - f(a)}{x - a}}_{\text{difference quotient.}}$$

IDEA: This definition captures the instantaneous rate of change * of f at a via the limit of average rates of change on ever-diminishing scales.

Ex: Let $f(x) = x^2 + 2x$. Calculate $f'(-1)$. $\quad a = -1$

Sol:

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{(x^2 + 2x) - ((-1)^2 + 2(-1))}{x + 1}$$

$$= \lim_{x \to -1} \frac{x^2 + 2x - (-1)}{x + 1}$$

$$= \lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x+1)}{\cancel{x+1}}$$

$$= \lim_{x \to -1} (x + 1) = (-1) + 1 = 0$$

$\boxed{/\!/}$

Ex: Calculate $f'(a)$ for $f(x) = 6x+3$ and $a = 2$.

Sol:
$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{x\to 2} \frac{(6x+3)-(6\cdot 2+3)}{x-2}$$
$$= \lim_{x\to 2} \frac{6x+3-15}{x-2}$$
$$= \lim_{x\to 2} \frac{6x-12}{x-2}$$
$$= \lim_{x\to 2} \frac{6\cancel{(x-2)}}{\cancel{x-2}}$$
$$= \lim_{x\to 2} 6 = 6 \quad \boxed{\S}$$

Ex: Calculate $f'(a)$ for $f(x) = \frac{1}{x}$. (IDEA: a is not known in advance)

Sol:
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$
$$= \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a}$$
$$= \lim_{x\to a} \frac{\frac{1}{x}\cdot\frac{a}{a}-\frac{1}{a}\cdot\frac{x}{x}}{x-a}$$
$$= \lim_{x\to a} \frac{\frac{a-x}{ax}}{x-a}$$
$$= \lim_{x\to a} \left(\frac{a-x}{ax}\cdot\frac{1}{x-a}\right)$$
$$= \lim_{x\to a} \left(\frac{-\cancel{(x-a)}^{1}}{ax}\cdot\frac{1}{\cancel{x-a}}\right)$$
$$= \lim_{x\to a} \frac{-1}{ax}$$
$$= \frac{-1}{a\cdot a} = -\frac{1}{a^2} \quad \boxed{\S}$$

(works if $a \neq 0$)

:)

Alternative Difference Quotient:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{[x - a]} = \lim_{a+h \to a} \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$[x = a + h]$

↑ near to a, h is near 0. *

Ex: Calculate $g'(a)$ for $g(x) = \sqrt{2x+1}$ and $a = \frac{1}{2}$.

Sol:

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(a+h)+1} - \sqrt{2 \cdot a + 1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(\frac{1}{2}+h)+1} - \sqrt{2 \cdot \frac{1}{2} + 1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1 + 2h + 1} - \sqrt{1+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2h+2} - \sqrt{2}}{h}$$

$$(x-y)(x+y) = x^2 - y^2$$

$$= \lim_{h \to 0} \frac{\sqrt{2h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{2h+2} + \sqrt{2}}{\sqrt{2h+2} + \sqrt{2}}$$

$$= \lim_{h \to 0} \frac{(2h+2) - 2}{h(\sqrt{2h+2} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2h+2} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2h+2} + \sqrt{2}} = \frac{2}{\sqrt{2 \cdot 0 + 2} + \sqrt{2}} = \frac{2}{\sqrt{2} + \sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \boxtimes$$

Ex: Calculate $f'(a)$ for $f(t) = t^2 - 2t + 11$ at $a = \pi$.

Sol: $$f'(\pi) = \lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h}$$

$$= \lim_{h\to 0} \frac{\left((\pi+h)^2 - 2(\pi+h) + 11\right) - \left(\pi^2 - 2\pi + 11\right)}{h}$$

$a^2 - b^2$
$= (a-b)(a+b)$

$$= \lim_{h\to 0} \frac{\left((\pi+h)^2 - \pi^2\right) - \left(2(\pi+h) - 2\pi\right) + (11 - 11)}{h}$$

$$= \lim_{h\to 0} \frac{(\pi+h-\pi)(\pi+h+\pi) - (2\pi + 2h - 2\pi) + 0}{h}$$

$$= \lim_{h\to 0} \frac{h(2\pi+h) - 2h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2\pi+h-2)}{\cancel{h}} = \lim_{h\to 0} (2\pi + h - 2) = 2\pi + 0 - 2 = 2\pi - 2 \quad \boxed{}$$

Ex: Calculate $f'(a)$ for $f(x) = x^3$. (a is indeterminate)

Sol: $$f'(a) = \left[\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right] = \underbrace{\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

$$= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h}$$

$$= \lim_{h\to 0} \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h}$$

$$= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3a^2 + 3ah + h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} (3a^2 + 3ah + h^2) = 3a^2 + 3a\cdot 0 + 0^2 = 3a^2 \quad \boxed{}$$

Side work

$$(a+h)^3 = (a+h)(a+h)^2$$
$$= (a+h)\underbrace{(a^2 + 2ah + h^2)}$$
$$= a(a^2 + 2ah + h^2) + h(a^2 + 2ah + h^2)$$
$$= a^3 + 2a^2h + ah^2 + a^2h + 2ah^2 + h^3$$
$$= a^3 + 3a^2h + 3ah^2 + h^3$$

$$b^3 - c^3 = (b-c)(b^2 + bc + c^2)$$

(check: $= b^3 + \cancel{b^2c} + \cancel{bc^2} - \cancel{cb^2} - \cancel{bc^2} - c^3$)
$= b^3 - c^3$

NB: if you used $\lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$, Ex: Calculate an equation of the tangent line to $f(x) = x + \sqrt{x}$ at $x = 1$.

Sol: $y - y_0 = m(x - x_0)$

Need: point (x_0, y_0)
slope $m = f'(1)$

point: $(1, f(1)) = (1, 1+\sqrt{1}) = (1, 2)$

slope: $m = f'(1) = \lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0} \dfrac{((1+h) + \sqrt{1+h}) - 2}{h}$

$= \lim_{h\to 0} \dfrac{h + \sqrt{1+h} - 1}{h}$

$= \lim_{h\to 0} \left(\dfrac{h}{h} + \dfrac{\sqrt{1+h} - 1}{h} \right)$

$= \lim_{h\to 0} \left(1 + \dfrac{\sqrt{1+h} - 1}{h} \cdot \dfrac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right)$

$= \lim_{h\to 0} \left(1 + \dfrac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \right)$

$= \lim_{h\to 0} \left(1 + \dfrac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + 1)} \right)$

$= \lim_{h\to 0} \left(1 + \dfrac{1}{\sqrt{1+h} + 1} \right)$

$= 1 + \dfrac{1}{\sqrt{1+0} + 1} = 1 + \dfrac{1}{2} = \dfrac{3}{2}$

$\therefore$ tan line has equation

$y - 2 = \frac{3}{2}(x - 1)$ $\boxed{//}$

Ex: Find the tangent line to $f(x) = \dfrac{1}{x+1}$ at $x = 2$.

Sol: $y - y_0 = m(x - x_0)$

Point: $(2, f(2)) = \left(2, \frac{1}{2+1}\right) = \left(2, \frac{1}{3}\right)$

Slope: $m = f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim_{h\to 0} \left(\dfrac{\frac{1}{2+h+1} - \frac{1}{3}}{h} \cdot \dfrac{3(h+3)}{3(h+3)} \right)$

$= \lim_{h\to 0} \dfrac{3 - (h+3)}{h \cdot 3 \cdot (h+3)}$

$= \lim_{h\to 0} \dfrac{-\cancel{h}\,1}{3\cancel{h}(h+3)}$

$= \lim_{h\to 0} \dfrac{-1}{3(h+3)}$

$= \dfrac{-1}{3(0+3)} = -\dfrac{1}{9}$

$\therefore$ the tangent line has equation

$y - \frac{1}{3} = -\frac{1}{9}(x - 2)$ $\boxed{//}$

Ex: Suppose a particle moves along the x-axis. Its position is given by the function $\boxed{S(t) = \frac{16}{t^2} - \frac{4}{t}}$. What is the...

① Velocity of the particle at time $t=2$ seconds?

② average velocity of the particle from $t=1$ to $t=3$?

Sol:

① $V(2) = S'(2) = \lim_{h\to 0} \frac{S(2+h)-S(2)}{h}$

$= \lim_{h\to 0} \frac{\left(\frac{16}{(2+h)^2} - \frac{4}{2+h}\right) - \left(\frac{16}{2^2} - \frac{4}{2}\right)}{h} \cdot \frac{(2+h)^2 2^2}{(2+h)^2 2^2}$

$= \lim_{h\to 0} \frac{\left(16\cdot 2^2 - 4\cdot 2^2 (2+h)\right) - \left(16(2+h)^2 - 4(2+h)^2 \cdot 2\right)}{h}$

$= \lim_{h\to 0} \frac{64 - 32 - 16h - (2+h)^2(16-8)}{h}$

$= \lim_{h\to 0} \frac{32 - 16h - 8(4+4h+h^2)}{h} = \lim_{h\to 0} \frac{\cancel{32} - 16h \cancel{-32} - 32h - 8h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(-48-8h)}{\cancel{h}} = \lim_{h\to 0} (-48-8h) = -48 - 8\cdot 0 = -48$

$\therefore$ The instantaneous velocity of the particle at $t=2$ is -48 units/s.

② Average velocity: $\frac{S(3)-S(1)}{3-1} = \frac{\left(\frac{16}{3^2} - \frac{4}{3}\right) - \left(\frac{16}{1^2} - \frac{4}{1}\right)}{2}$

$= \frac{\frac{16}{9} - \frac{12}{9} - 12}{2}$

$= \frac{\frac{4}{9} - \frac{108}{9}}{2} = -\frac{104}{9} \cdot \frac{1}{2} = -\frac{52}{9}$

$\therefore$ the average velocity of the partical on this interval is $-\frac{52}{9}$ units/sec. ∎