

COMPUTING DERIVATIVES

Recall: for function f defined on an interval containing a ,

The derivative of f at a is (provided it exists):

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad \text{difference quotient.}$$

IDEA: This definition captures the instantaneous rate of change of f at a via the limit of average rates of change on ever-diminishing scales.

Ex: Let $f(x) = x^2 + 2x$. Calculate $f'(-1)$.

$a = -1$

Sol:

$$\begin{aligned} \underline{f'(-1)} &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x \rightarrow -1} \frac{(x^2 + 2x) - ((-1)^2 + 2(-1))}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{x^2 + 2x - (-1)}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{\cancel{(x+1)}(x+1)}{\cancel{x+1}} \\ &= \lim_{x \rightarrow -1} (x+1) = (-1) + 1 = \underline{0} \end{aligned}$$

\square

Ex: Calculate $f'(a)$ for $f(x) = 6x + 3$ and $a = 2$.

Sol:

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(6x + 3) - (6 \cdot 2 + 3)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{6x + 3 - 15}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{6x - 12}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{6(x - 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} 6 = 6 \quad \square \end{aligned}$$

Ex: Calculate $f'(a)$ for $f(x) = \frac{1}{x}$.

(IDEA: a is not known in advance)

Sol:

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{x} \cdot \frac{a}{a} - \frac{1}{a} \cdot \frac{x}{x}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{a - x}{ax}}{x - a} \\ &= \lim_{x \rightarrow a} \left(\frac{a - x}{ax} \cdot \frac{1}{x - a} \right) \\ &= \lim_{x \rightarrow a} \left(\frac{-(x - a)}{ax} \cdot \frac{1}{x - a} \right) \end{aligned}$$

$\rightarrow = \lim_{x \rightarrow a} \frac{-1}{ax}$

$$= -\frac{1}{a \cdot a} = -\frac{1}{a^2} \quad \square$$

(works if $a \neq 0$)
☺

Alternative Difference Quotient:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{a+h \rightarrow a} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$[x = a+h]$ \uparrow near to a , h is near 0 . *

Ex: Calculate $g'(a)$ for $g(x) = \sqrt{2x+1}$ and $a = \frac{1}{2}$.

Sol:

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\sqrt{2(a+h)+1} - \sqrt{2a+1}}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\sqrt{2(\frac{1}{2}+h)+1} - \sqrt{2 \cdot \frac{1}{2} + 1}}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\sqrt{1+2h+1} - \sqrt{1+1}}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\sqrt{2h+2} - \sqrt{2}}{h}$$

$(x-y)(x+y) = x^2 - y^2$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{2h+2} + \sqrt{2}}{\sqrt{2h+2} + \sqrt{2}}$$
$$= \lim_{h \rightarrow 0} \frac{(2h+2) - 2}{h(\sqrt{2h+2} + \sqrt{2})}$$
$$= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2h+2} + \sqrt{2})}$$
$$= \lim_{h \rightarrow 0} \frac{2}{\sqrt{2 \cdot 0 + 2} + \sqrt{2}} = \frac{2}{\sqrt{2} + \sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \square$$

Ex: Calculate $f'(a)$ for $f(t) = t^2 - 2t + 11$ at $a = \pi$.

Sol:

$$f'(\pi) = \lim_{h \rightarrow 0} \frac{f(\pi+h) - f(\pi)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\pi+h)^2 - 2(\pi+h) + 11 - (\pi^2 - 2\pi + 11)}{h}$$

$a^2 - b^2 = (a-b)(a+b)$

$$= \lim_{h \rightarrow 0} \frac{(\pi+h)^2 - \pi^2 - (2(\pi+h) - 2\pi) + (11-11)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\pi+h-\pi)(\pi+h+\pi) - (2\pi+2h-2\pi) + 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2\pi+h) - 2h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2\pi+h-2)}{h} = \lim_{h \rightarrow 0} (2\pi+h-2) = 2\pi+0-2 = 2\pi-2$$

Ex: Calculate $f'(a)$ for $f(x) = x^3$. (a is indeterminate)

Sol:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(3a^2 + 3ah + h^2)}{h}$$

$$= \lim_{h \rightarrow 0} (3a^2 + 3ah + h^2) = 3a^2 + 3a \cdot 0 + 0^2 = 3a^2$$

Side work

$$(a+h)^3 = (a+h)(a+h)^2$$

$$= (a+h)(a^2 + 2ah + h^2)$$

$$= a(a^2 + 2ah + h^2) + h(a^2 + 2ah + h^2)$$

$$= a^3 + 2a^2h + ah^2 + a^2h + 2ah^2 + h^3$$

$$= a^3 + 3a^2h + 3ah^2 + h^3$$

NB: if you used

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$b^3 - c^3 = (b-c)(b^2 + bc + c^2)$$

(check: $= b^3 + b^2c + bc^2 - c^3 - bc^2 - c^3 = b^3 - c^3$)

Ex: Calculate an equation of the tangent line to $f(x) = \underline{x + \sqrt{x}}$ at $x = 1$.

Sol: $y - y_0 = m(x - x_0)$

Need: point (x_0, y_0)
slope $m = f'(1)$

Point: $(1, f(1)) = (1, 1 + \sqrt{1}) = (1, 2)$

Slope: $m = f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \rightarrow 0} \frac{((1+h) + \sqrt{1+h}) - 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h + \sqrt{1+h} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{h}{h} + \frac{\sqrt{1+h} - 1}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(1 + \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right)$$

$$= \lim_{h \rightarrow 0} \left(1 + \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \right)$$

$$= \lim_{h \rightarrow 0} \left(1 + \frac{h}{h(\sqrt{1+h} + 1)} \right)$$

$$= \lim_{h \rightarrow 0} \left(1 + \frac{1}{\sqrt{1+h} + 1} \right)$$

$$= 1 + \frac{1}{\sqrt{1+0} + 1} = 1 + \frac{1}{2} = \frac{3}{2}$$

\therefore tan line has equation

$$y - 2 = \frac{3}{2}(x - 1) \quad \square$$

Ex: Find the tangent line to $f(x) = \frac{1}{x+1}$ at $x = 2$.

Sol: $y - y_0 = m(x - x_0)$

Point: $(2, f(2)) = (2, \frac{1}{2+1}) = (2, \frac{1}{3})$

Slope: $m = f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \rightarrow 0} \left(\frac{\frac{1}{2+h+1} - \frac{1}{3}}{h} \cdot \frac{3(h+3)}{3(h+3)} \right)$$

$$= \lim_{h \rightarrow 0} \frac{3 - (h+3)}{h \cdot 3 \cdot (h+3)}$$

$$= \lim_{h \rightarrow 0} \frac{-h}{3h(h+3)}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{3(h+3)}$$

$$= \frac{-1}{3(0+3)} = -\frac{1}{9}$$

\therefore the tangent line has equation

$$y - \frac{1}{3} = -\frac{1}{9}(x - 2) \quad \square$$

Ex: Suppose a particle moves along the x-axis. Its position is given by the function $S(t) = \frac{16}{t^2} - \frac{4}{t}$. What is the...

- ① Velocity of the particle at time $t=2$ seconds?
- ② average velocity of the particle from $t=1$ to $t=3$?

Sol:

$$\begin{aligned} \textcircled{1} \quad V(2) &= S'(2) = \lim_{h \rightarrow 0} \frac{S(2+h) - S(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{16}{(2+h)^2} - \frac{4}{2+h}\right) - \left(\frac{16}{2^2} - \frac{4}{2}\right)}{h} \cdot \frac{(2+h)^2 \cdot 2^2}{(2+h)^2 \cdot 2^2} \\ &= \lim_{h \rightarrow 0} \frac{(16 \cdot 2^2 - 4 \cdot 2^2 \cdot (2+h)) - (16(2+h)^2 - 4(2+h)^2 \cdot 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{64 - 32 - 16h - (2+h)^2(16-8)}{h} \\ &= \lim_{h \rightarrow 0} \frac{32 - 16h - 8(4 + 4h + h^2)}{h} = \lim_{h \rightarrow 0} \frac{\cancel{32} - 16h - \cancel{32} - 32h - 8h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-48 - 8h}{1} = \lim_{h \rightarrow 0} (-48 - 8h) = -48 - 8 \cdot 0 = -48 \end{aligned}$$

\therefore The instantaneous velocity of the particle at $t=2$ is -48 units/s.

$$\begin{aligned} \textcircled{2} \quad \text{Average velocity:} \quad \frac{S(3) - S(1)}{3 - 1} &= \frac{\left(\frac{16}{3^2} - \frac{4}{3}\right) - \left(\frac{16}{1^2} - \frac{4}{1}\right)}{2} \\ &= \frac{\frac{16}{9} - \frac{12}{9} - 12}{2} \\ &= \frac{\frac{4}{9} - \frac{108}{9}}{2} = \frac{-104}{9} \cdot \frac{1}{2} = -\frac{52}{9} \end{aligned}$$

\therefore the average velocity of the particle on this interval is $-\frac{52}{9}$ units/sec.