

CONTINUITY

IDEA: We want to study functions which are "unbroken."

Intuitively, these are the functions which can be drawn without picking up your pencil.

Defⁿ: A function f is continuous at a when

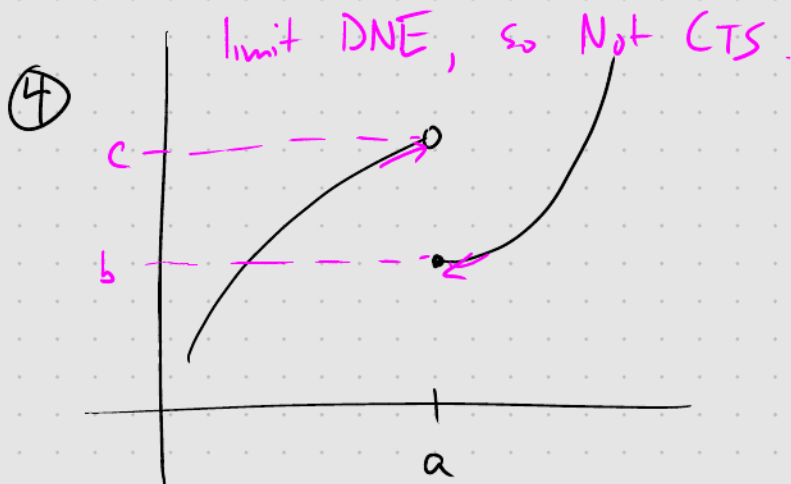
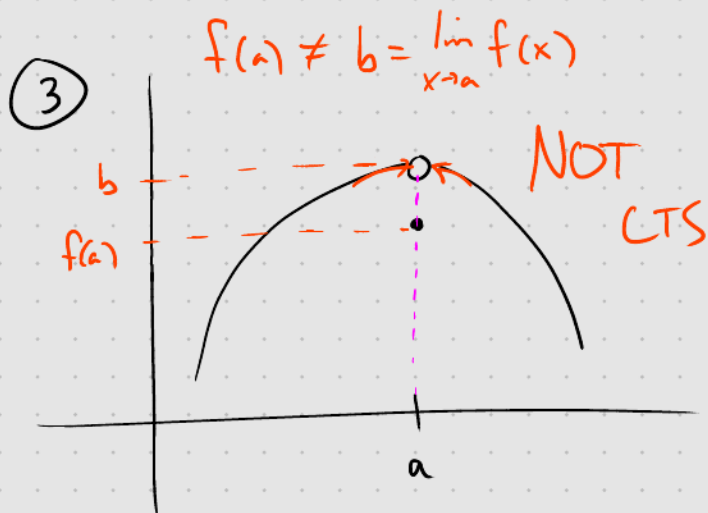
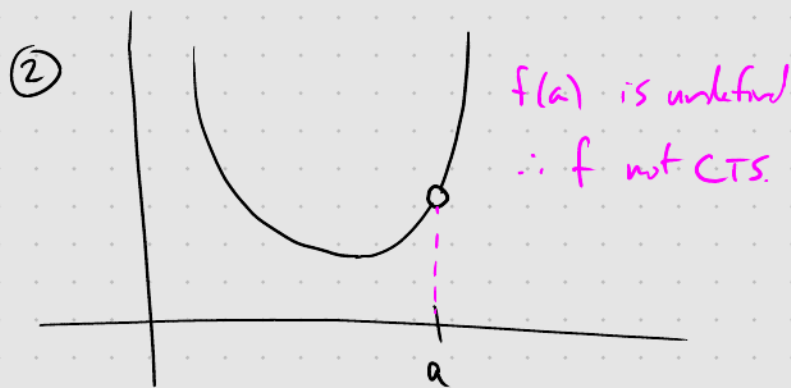
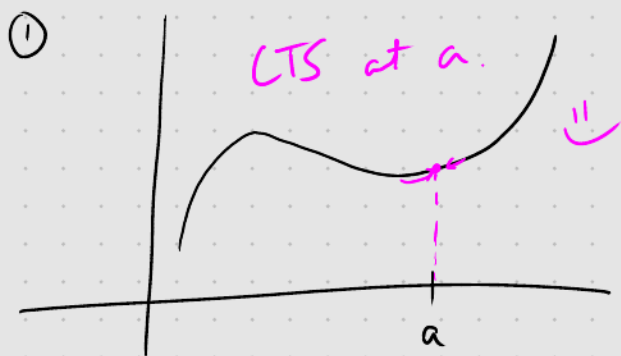
$$\lim_{x \rightarrow a^-} f(x) = \underline{f(a)} = \lim_{x \rightarrow a^+} f(x).$$

*observe: ① limit exists
② function defined at a .*

Note: I'll usually abbreviate "continuous" to "cts".

The definition forces the limit $\lim_{x \rightarrow a} f(x) = \underline{f(a)}$ to exist.

Examples and nonexamples



Ex: Is $f(x) = \frac{x^2 - x - 2}{x - 2}$ continuous at $x = 2$?

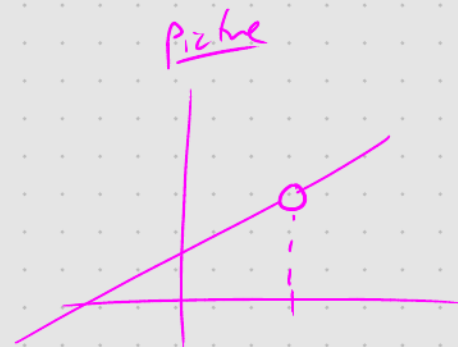
Sol: (Recall: $\lim_{x \rightarrow a} f(x) = f(a)$)

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \rightarrow 2} \frac{(x+1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x \rightarrow 2} (x+1) = 2+1 = 3$$

$f(2)$ is UNDEFINED

$\therefore f$ is NOT CTS at 2.



□

Ex: Is $f(x) = x^2 - 2x + 5$ continuous at 3?

Sol: $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (x^2 - 2x + 5)$ ↙

$$= 3^2 - 2 \cdot 3 + 5$$

$$= 9 - 6 + 5$$

$$= 8$$

$$f(3) = 3^2 - 2 \cdot 3 + 5 = 8$$

$\therefore \lim_{x \rightarrow 3} f(x) = f(3)$, so f is cts at 3.

□

Ex: Consider $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ ←

Is $f(x)$ cts at 0?

Sol: $f(0) = 1$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$, so f is cts at 0. \square

Ex: Consider $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$

Is $f(x)$ cts at $x=1$?



Sol: $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x+1) = 2 \cdot 1 + 1 = 3$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (-x+4) = -1 + 4 = 3$$

$f(1) = 2 \neq 3 = \lim_{x \rightarrow 1^+} f(x)$ $\therefore f$ is not cts at 1. \square

Prop: The following are continuous at all points in their domains:

- ① polynomials,
- ② rational functions,
- ③ trigonometric functions.

Ex: Where is $f(x) = \begin{cases} 5+x^3 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 2-x & \text{if } 1 \leq x \end{cases}$ cts?



Sol: B/c polys are cts everywhere, we have
f is cts (at least) everywhere $x \neq -2, x \neq 1$.

Check $x = -2$

$$f(-2) = 5 + (-2)^3 = 5 - 8 = -3$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (5 + x^3) = 5 + (-2)^3 = 5 - 8 = -3$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} x^2 = (-2)^2 = 4$$

$$\therefore \lim_{x \rightarrow -2} f(x) \text{ DNE}$$

\therefore f is not cts at $x = -2$

Check $x = 1$

$$f(1) = 2 - 1 = 1; \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1^2 = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x) = 2 - 1 = 1$$

$\therefore \lim_{x \rightarrow 1} f(x) = 1 = f(1)$. \therefore f is cts at $x = 1$.

\therefore f is cts on $(-\infty, -2) \cup (-2, \infty)$



Ex: Where is $h(x) = \begin{cases} 4x+6 & \text{if } x \leq -3 \\ 3-x^2 & \text{if } -3 < x \leq 0 \\ x+5 & \text{if } 0 < x < 1 \\ x^2-2x+7 & \text{if } 1 < x \end{cases}$ cts?

Sol: 

Obs: all pieces are poly.
we'll analyze breakpoints.

* @ $x=1$: f is undefined NOT CTS

* @ $x=0$: $f(0) = 3 - 0^2 = 3$ ^{NOT eq. l} > NOT CTS

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x+5) = 0+5 = 5$$

@ $x=-3$: $f(-3) = 4(-3) + 6 = -12 + 6 = -6$

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (3-x^2) = 3 - (-3)^2 = 3 - 9 = -6$$

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (4x+6) = 4(-3) + 6 = -12 + 6 = -6$$

$\therefore f$ is cts at $x=-3$.

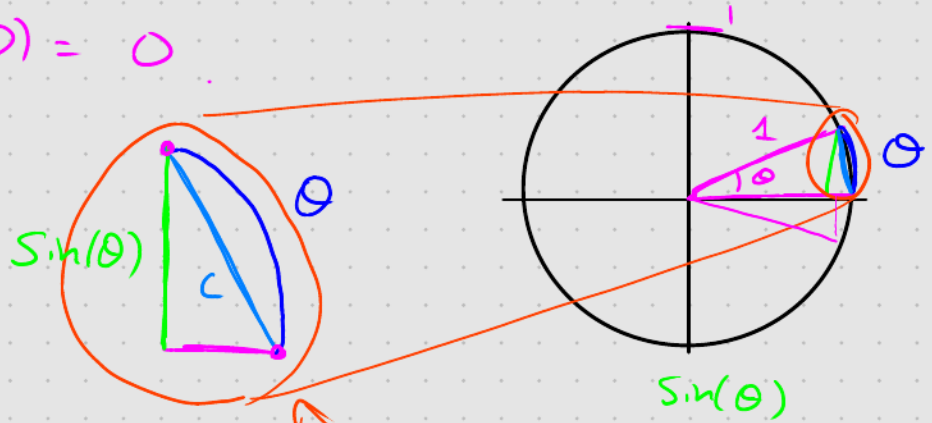
Hence f is cts on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ \square

CONTINUITY OF SINE AND COSINE

Goal: Show that $\sin(x)$ and $\cos(x)$ are cts on $(-\infty, \infty)$.

Step 1: $\sin(\theta)$ and $\cos(\theta)$ are cts at 0.

WTS: $\lim_{\theta \rightarrow 0} \sin(\theta) = \sin(0) = 0$.



$$\sin(\theta) \leq \theta \leq \theta$$

So for $\theta \geq 0$,

$$0 \leq \sin(\theta) \leq \theta$$

if $\theta < 0$, this flips

$$\therefore 0 = \lim_{\theta \rightarrow 0^+} 0 \leq \lim_{\theta \rightarrow 0^+} \sin(\theta) \leq \lim_{\theta \rightarrow 0^+} \theta = 0$$

$\therefore \lim_{\theta \rightarrow 0^+} \sin(\theta) = 0$ by the Squeeze Thm.

On the other hand, if $\theta < 0$, then $\theta \leq \sin(\theta) \leq 0$, we again apply the Squeeze Thm:

$$\therefore 0 = \lim_{\theta \rightarrow 0^-} \theta \leq \lim_{\theta \rightarrow 0^-} \sin(\theta) \leq \lim_{\theta \rightarrow 0^-} 0 = 0$$

$\therefore \lim_{\theta \rightarrow 0^-} \sin(\theta) = 0$. To summarize:

$\lim_{\theta \rightarrow 0} \sin(\theta) = 0 = \sin(0)$, so $\sin(\theta)$ is cts at 0.

For $\cos(\theta)$, if θ is close to 0, so b/c

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \text{or} \quad \cos(\theta) \geq 0, \text{ we have}$$

$$\cos(\theta) = \sqrt{1 - \sin^2(\theta)} \quad \therefore \lim_{\theta \rightarrow 0} \cos(\theta) = \lim_{\theta \rightarrow 0} \sqrt{1 - \sin^2(\theta)} = 1 = \cos(0) \quad \square$$

Step 2: Apply trigonometric identities

★ Recall: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. ★

Want to show: for all a , $\lim_{\theta \rightarrow a} \sin(\theta) = \sin(a)$.

$$\begin{aligned} \lim_{\theta \rightarrow a} \sin(\theta) &= \lim_{\theta \rightarrow a} \sin((\theta-a) + a) \\ &= \lim_{\theta \rightarrow a} (\sin(\theta-a)\cos(a) + \cos(\theta-a)\sin(a)) \end{aligned}$$

$$\begin{aligned} &= \cos(a) \lim_{\theta \rightarrow a} \sin(\theta-a) + \sin(a) \lim_{\theta \rightarrow a} \cos(\theta-a) \\ \Leftrightarrow \theta \rightarrow a \Rightarrow \theta-a \rightarrow 0 &\rightarrow = \cos(a) \lim_{\theta-a \rightarrow 0} \sin(\theta-a) + \sin(a) \lim_{\theta-a \rightarrow 0} \cos(\theta-a) \end{aligned}$$

$$\text{let } y = \theta-a \rightarrow = \cos(a) \lim_{y \rightarrow 0} \sin(y) + \sin(a) \lim_{y \rightarrow 0} \cos(y)$$

$$\begin{aligned} \text{b/c } \sin(\theta) \text{ and } \cos(\theta) \text{ are cts at } 0 &\rightarrow = \cos(a) \sin(0) + \sin(a) \cos(0) \\ &= \cos(a) \cdot 0 + \sin(a) \cdot 1 \\ &= \sin(a) \end{aligned}$$

$\therefore \sin(\theta)$ is cts at all a .

The calculation for $\cos(\theta)$ is similar:

Use the identity: $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Take-away: $\sin(\theta)$ and $\cos(\theta)$ are cts functions of θ on $(-\infty, \infty)$.

PROPERTIES OF CONTINUITY

GOAL: Use limit laws to investigate continuity.

Prop: Suppose f and g are cts at a .

① $f+g$ is cts at a .

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \stackrel{f, g \text{ cts}}{=} f(a) + g(a)$$

② $f \cdot g$ is cts at a .

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right) \stackrel{f, g \text{ cts}}{=} f(a) \cdot g(a)$$

③ If $g(a) \neq 0$, then f/g is cts at a .

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \stackrel{\substack{\uparrow \\ \text{if } \lim_{x \rightarrow a} g(x) \neq 0}}{=} \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \stackrel{\substack{\uparrow \\ f, g \text{ cts}}}{=} \frac{f(a)}{g(a)} \quad \leftarrow \text{NOT } 0 \text{ by assumption } \Downarrow$$

③ If f is cts at $g(a)$, then $f \circ g$ is cts at a .

$$\begin{aligned} \lim_{x \rightarrow a} (f \circ g)(x) &= \lim_{x \rightarrow a} f(g(x)) = \lim_{\substack{g(x) \rightarrow g(a) \\ \text{exactly when } \\ g(x) \rightarrow g(a) \text{ b/c } g \text{ is cts at } a}} f(g(x)) \\ &= \lim_{y \rightarrow g(a)} f(y) \quad \leftarrow y = g(x) \\ &= f(g(a)) \quad \leftarrow f \text{ is cts at } g(a) \end{aligned}$$

TAKE-AWAY: The usual operations on functions behave predictably with respect to continuity.

The following are all continuous (on their domain)

[* polys] [* rats] [* trig functions] [* roots]

$+$, \cdot , \div
and comp.