CONTINUITY

IDEA: We want to study functions which are "unbroken."
Intuitively, these are the functions which can be drawn without picking up your pencil.

Def^n: A function f is continuous at a when

$$\underbrace{\lim_{x\to a^-} f(x)} = \underline{f(a)} = \underbrace{\lim_{x\to a^+} f(x)}.$$

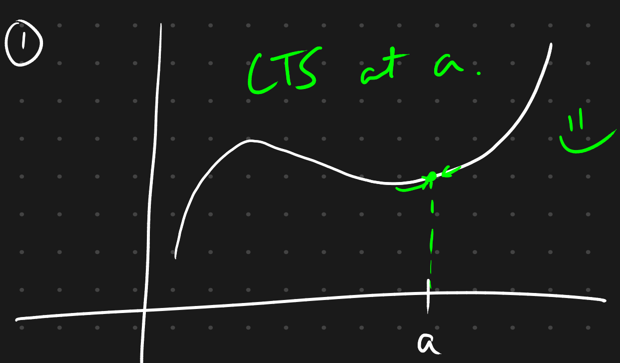
observe: ① limit exists
② function defined at a.

Note: I'll usually abbreviate "continuous" to "cts".

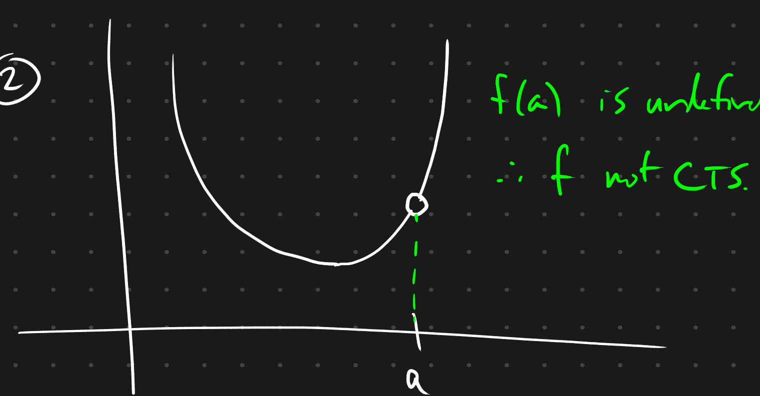
The definition forces the limit $\lim_{x\to a} f(x) = f(a)$ to exist.

Examples and nonexamples

①



②



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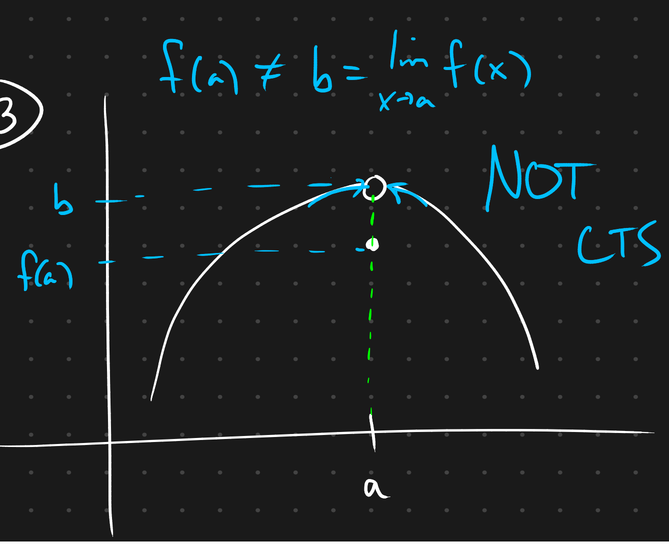
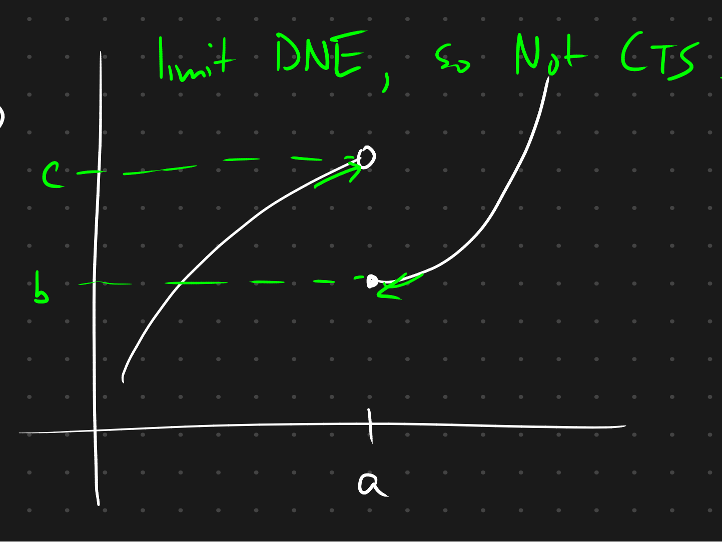


④



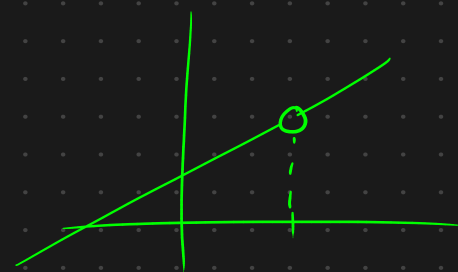
Ex: Is $f(x) = \frac{x^2-x-2}{x-2}$ Continuous at $x=2$?

Sol: (Recall: $\lim_{x \to a} f(x) = f(a)$)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x+1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x \to 2} (x+1) = 2+1 = 3$$

Picture



$f(2)$ is UNDEFINED

$\therefore$ f is NOT CTS at 2. $\boxed{\checkmark}$

Ex: Is $f(x) = x^2-2x+5$ Continuous at 3?

Sol: $\lim_{x\to 3} f(x) = \lim_{x \to 3} (x^2-2x+5)$

$$= 3^2 - 2\cdot 3 + 5$$

$$= 9-6+5$$

$$= 8$$

$f(3) = 3^2 - 2\cdot 3 + 5 = 8$

$\therefore$ $\lim_{x\to 3} f(x) = f(3)$, so f is cts at 3. $\boxed{\checkmark}$

Ex: Consider $f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \leftarrow \\ 1 & \text{if } x = 0 \end{cases}$

Is $f(x)$ cts at 0?

Sol: $f(0) = 1$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$\therefore \lim_{x \to 0} f(x) = f(0)$, so f is cts at 0. $\boxed{\text{Ġ}}$

Ex: Consider $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$.

Is $f(x)$ cts at $x=1$?

Sol: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x+1) = 2 \cdot 1 + 1 = 3$

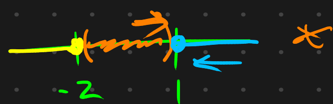
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x+4) = -1 + 4 = 3$

$f(1) = 2 \neq 3 = \lim_{x \to 1^+} f(x)$ $\therefore f$ is not cts at 1. $\boxed{\text{Ġ}}$

Prop: The following are continuous at all points in their domains:

① polynomials, ② rational functions, ③ trigonometric functions.

Ex: Where is $f(x) = \begin{cases} 5 + x^3 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 2 - x & \text{if } 1 \leq x \end{cases}$ cts?



Sol: B/c polys are cts everywhere, we have f is cts (at least) everywhere $x \neq -2$, $x \neq 1$.

Check $x = -2$

$f(-2) = 5 + (-2)^3 = 5 - 8 = -3$

$\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2} (5 + x^3) = 5 + (-2)^3 = 5 - 8 = -3$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} x^2 = (-2)^2 = 4$

$\therefore \lim\limits_{x \to -2} f(x)$ DNE

$\therefore f$ is not cts at $x = -2$

Check $x = 1$

$f(1) = 2 - 1 = 1$; $\quad \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1^2 = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 2 - 1 = 1$

$\therefore \lim\limits_{x \to 1} f(x) = 1 = f(1)$. $\therefore f$ is cts at $x = 1$.

$\therefore f$ is cts on $(-\infty, -2) \cup (-2, \infty)$ $\boxed{\checkmark}$

Ex: Where is $h(x) = \begin{cases} 4x+6 & \text{if } x \le -3 \\ 3-x^2 & \text{if } -3 < x \le 0 \\ x+5 & \text{if } 0 < x < 1 \\ x^2-2x+7 & \text{if } 1 < x \end{cases}$ cts?

Sol:

Obs: all pieces are poly. We'll analyze breakpoints.

* @ $x=1$: f is undefined NOT CTS

* @ $x=0$: $f(0) = 3 - 0^2 = 3$ NOT equal $\Rightarrow$ NOT CTS

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+5) = 0+5 = 5$

@ $x=-3$: $f(-3) = 4(-3)+6 = -12+6 = -6$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (3-x^2) = 3-(-3)^2 = 3-9 = -6.$

$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (4x+6) = 4(-3)+6 = -12+6 = -6.$

$\therefore$ f is cts at $x=-3$.

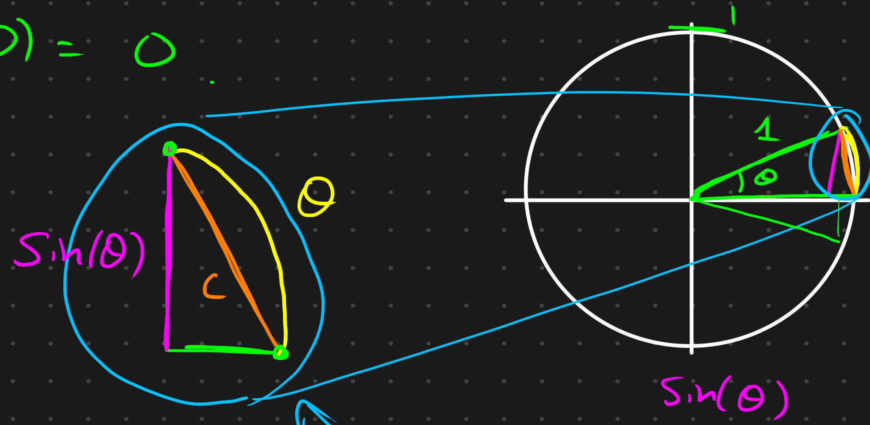
Hence f is cts on $(-\infty, 0) \cup (0,1) \cup (1, \infty)$ $\boxtimes$

Continuity of Sine and Cosine

Goal: Show that $\sin(x)$ and $\cos(x)$ are cts on $(-\infty,\infty)$.

Step 1: $\sin(\theta)$ and $\cos(\theta)$ are cts at 0.

WTS: $\lim_{\theta\to 0}\sin(\theta)=\sin(0)=0$



if $\theta<0$, this flips

$\sin(\theta)\le C\le\theta$

So for $\theta\ge 0$,

$$\boxed{0\le\sin(\theta)\le\theta}$$

$\therefore\ 0=\lim_{\theta\to 0^+}0\le\lim_{\theta\to 0^+}\sin(\theta)\le\lim_{\theta\to 0^+}\theta=0$

$\therefore\ \lim_{\theta\to 0^+}\sin(\theta)=0$ by the Squeeze Thm.

On the other hand, if $\theta<0$, then

$\theta\le\sin(\theta)\le 0$, we again apply the Squeeze Thm:

$\therefore\ 0=\lim_{\theta\to 0^-}\theta\le\lim_{\theta\to 0^-}\sin(\theta)\le\lim_{\theta\to 0^-}0=0$

$\therefore\ \lim_{\theta\to 0^-}\sin(\theta)=0$. To summarize:

$\lim_{\theta\to 0}\sin(\theta)=0=\sin(0)$, so $\sin(\theta)$ is cts at 0.

For $\cos(\theta)$, if θ is close to 0, so b/c

$\sin^2(\theta)+\cos^2(\theta)=1$ and $\cos(\theta)\ge 0$, we have

$\cos(\theta)=\sqrt{1-\sin^2(\theta)}$ $\therefore\ \lim_{\theta\to 0}\cos(\theta)=\lim_{\theta\to 0}\sqrt{1-\sin^2(\theta)}=1=\cos(0)$ $\blacksquare$

Step 2: Apply trigonometric identities

☆ Recall: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. ☆

Want to show: for all a, $\lim_{\theta\to a}\sin(\theta) = \sin(a)$.

$$\lim_{\theta\to a}\sin(\theta) = \lim_{\theta\to a}\sin((\theta-a)+a)$$

$$= \lim_{\theta\to a}\left(\sin(\theta-a)\cos(a) + \cos(\theta-a)\sin(a)\right)$$

$$= \cos(a)\lim_{\theta\to a}\sin(\theta-a) + \sin(a)\lim_{\theta\to a}\cos(\theta-a)$$

$\theta\to a \iff \theta-a\to 0$

$$= \cos(a)\lim_{\theta-a\to 0}\sin(\theta-a) + \sin(a)\lim_{\theta-a\to 0}\cos(\theta-a)$$

let $y=\theta-a$

$$= \cos(a)\lim_{y\to 0}\sin(y) + \sin(a)\lim_{y\to 0}\cos(y)$$

b/c $\sin(\theta)$ and $\cos(\theta)$ are cts at 0

$$= \cos(a)\sin(0) + \sin(a)\cos(0)$$

$$= \cos(a)\cdot 0 + \sin(a)\cdot 1$$

$$= \sin(a)$$

$\therefore$ $\sin(\theta)$ is cts at all a.

The calculation for $\cos(\theta)$ is similar:

Use the identity: $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Take-away: $\sin(\theta)$ and $\cos(\theta)$ are cts functions of θ on $(-\infty,\infty)$.

PROPERTIES OF CONTINUITY

GOAL: Use limit laws to investigate continuity.

Prop: Suppose f and g are cts at a.

① $f+g$ is cts at a.

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \overset{f,g \text{ cts}}{=} f(a)+g(a)$$

② $f\cdot g$ is cts at a.

$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) \overset{f,g \text{ cts}}{=} f(a)\cdot g(a)$$

③ If $g(a)\neq 0$, then f/g is cts at a.

$$\lim_{x\to a}\frac{f(x)}{g(x)} \underset{\text{if } \lim_{x\to a} g(x)\neq 0}{=} \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \underset{f,g \text{ cts}}{=} \frac{f(a)}{\underbrace{g(a)}_{\text{NOT 0 by assumption}}}$$

③ If f is cts at $g(a)$, then $f\circ g$ is cts at a.

$$\lim_{x\to a}(f\circ g)(x) = \lim_{\underbrace{x\to a}} f(g(x)) = \lim_{\underbrace{g(x)\to g(a)}_{y=g(x)}} f(g(x)) = \lim_{y\to g(a)} f(y) \underset{f \text{ is cts at } g(a)}{=} f(g(a))$$

$x\to a$ exactly when $g(x)\to g(a)$ b/c g is cts at a.

TAKE-AWAY: The usual operations on functions behave predictably with respect to continuity. $+, \cdot, \div$ and comp.

The following are all continuous (on their domain)

[* polys] [* rats] [* trig functions] [* roots]