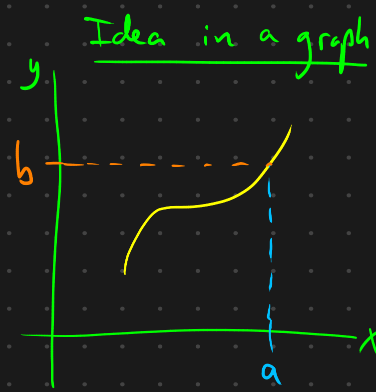
Limit Laws

Goal: We have an intuitive idea of limit. We want rules for computation.



Limit Idea: $\lim_{x \to a} f(x) = b$ is shorthand for "as x approaches a, $f(x)$ approaches b"

Prop (Limit Laws): Let f and g be functions and let a and c be real numbers. Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

⓪ $\lim_{x \to a} c = c$ (Constant Law)

① $\lim_{x \to a} x = a$ (Limit of x)

② $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ (Sum Law)

③ $\lim_{x \to a} \left(f(x) \cdot g(x) \right) = \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} g(x) \right)$ (Product Law)

④ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$. (Quotient Law)

NB: We can combine the rules above to get new rules.

⑤ $\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x)$ (Constant Multiple Law)

⑥ $\lim_{x \to a} \left(f(x) - g(x) \right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ (Difference Law)

⑦ $\lim_{x \to a} \left(f(x) \right)^n = \left(\lim_{x \to a} f(x) \right)^n$ (Power Law)

Final Note: All algebra rules you know can be applied inside the limit!

Ex: Calculate $\lim_{x\to -3}(x^2-2x+42)$.

Sol: $\lim_{x\to -3}(x^2-2x+42)$

$= \lim_{x\to -3}(x^2) - \lim_{x\to -3}(2x) + \lim_{x\to -3} 42$ (Sum/Difference Law)

$= \left(\lim_{x\to -3} x\right)^2 - \lim_{x\to -3}(2x) + \lim_{x\to -3} 42$ (Power Law)

$= \left(\lim_{x\to -3} x\right)^2 - 2\left(\lim_{x\to -3} x\right) + \lim_{x\to -3} 42$ (Constant Multiple Law)

$= (-3)^2 - 2(-3) + \lim_{x\to -3} 42$ (Limit of x)

$= (-3)^2 - 2(-3) + 42$ (Constant Law)

$= 9 + 6 + 42 = 15 + 42 = 57$ (Arithmetic)

NOTE: Every polynomial is a sum of constant multiples of powers of x, the method we just used will always work. In other words, we can always do the following in sequence:

① Apply the Sum Law: $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

② Apply the Constant Multiple Law: $\lim_{x\to a}(c f(x)) = c \lim_{x\to a} f(x)$

③ Apply the Power Law: $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

④ Apply $\lim_{x\to a} x = a$, and ⑤ simplify the result

BUT, that has the same effect as just evaluating the poly. at a, So we usually short-circuit that work by evaluating instead.

Ex: Calculate $\lim_{x\to 2}(x^3-5x^2+7x-6)$.

Sol: x^3-5x^2+7x-6 is a poly, so just evaluate!

$\lim_{x\to 2}(x^3-5x^2+7x-6) = (2)^3 - 5(2)^2 + 7(2) - 6 = 8 - 20 + 14 - 6 = -4$ $\boxed{//}$

"plugged in" $x=2$

Ex: Calculate $\lim_{x \to 1} \frac{x^2+3x-2}{x+1}$.

Rat function $= \frac{\text{Poly}}{\text{poly}}$

Sol: $\lim_{x \to 1} \frac{x^2+3x-2}{x+1}$

$$\overset{QL}{=} \frac{\lim_{x \to 1} (x^2+3x-2)}{\lim_{x \to 1} (x+1)}$$

if $\lim_{x \to 1} (x+1) \neq 0$

$$= \frac{(1)^2+3(1)-2}{(1)+1}$$

evaluation trick

$$= \frac{1+3-2}{1+1} = \frac{2}{2} = 1$$

Ex: Calculate $\lim_{x \to -1} \frac{x^2-2x}{x+3}$.

Sol: $\lim_{x \to -1} \frac{x^2-2x}{x+3}$

$$\overset{QL}{=} \frac{\lim_{x \to -1} (x^2-2x)}{\lim_{x \to -1} (x+3)}$$

$$= \frac{(-1)^2-2(-1)}{(-1)+3}$$

$$= \frac{1+2}{2} = \frac{3}{2}$$

Ex: Calculate $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Sol: Note $\lim_{x\to 2}(x-2) = 0$... So QL doesn't apply...

and $\lim_{x\to 2}(x^2-4) = 2^2-4 = 0$.

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{x^2-2^2}{x-2}$$

$a^2-b^2 = (a+b)(a-b)$

check

$$= \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x\to 2}(x+2) = (2)+2 = 4$$

$\boxed{\checkmark}$

Takeaway: Through algebra, we can calculate here!

Ex: Calculate $\lim_{x\to -3} \frac{x+3}{x^2+x-6}$.

Sol: $\lim_{x\to -3} \frac{x+3}{x^2+x-6}$

$\lim_{x\to -3}(x+3) = (-3)+3 = 0$

$\lim_{x\to -3}(x^2+x-6) = (-3)^2+(-3)-6 = 9-9 = 0$

x^2+x-6
$= (x-2)(x+3)$

(check by distribution!)

a, b
$a+b = 1$, $\underline{a\cdot b = -6}$

$$= \lim_{x\to -3} \frac{1\cancel{(x+3)}}{(x-2)\cancel{(x+3)}}$$

$$= \lim_{x\to -3} \frac{1}{x-2}$$

$$\overset{QL}{=} \frac{\lim_{x\to -3} 1}{\lim_{x\to -3}(x-2)}$$

$$= \frac{1}{(-3)-2} = \frac{1}{-5} = -\frac{1}{5}$$

$\boxed{\checkmark}$

Problems: Calculate each of the following

(A) $\lim_{x\to -7} \frac{2x+14}{x^2+10x+21}$ (B) $\lim_{x\to -3} \frac{x^2+4x+3}{x^2+x-6}$ (C) $\lim_{x\to 1} \frac{x^2-6x+5}{x^2+2x-3}$

Ex: Calculate $\lim\limits_{x\to -4} \dfrac{\frac{6}{x+2}+3}{x+4}$.

Sol: $\lim\limits_{x\to -4} \dfrac{\frac{6}{x+2}+3}{x+4}$

Note: Can't apply QL...

$$= \lim_{x\to -4} \frac{\frac{6}{x+2} + 3\cdot\frac{x+2}{x+2}}{x+4}$$

dist.

$$= \lim_{x\to -4} \frac{\frac{6+3(x+2)}{x+2}}{x+4}$$

$$= \lim_{x\to -4} \frac{\frac{3x+12}{x+2}}{\frac{x+4}{1}}$$

$$= \lim_{x\to -4} \frac{3x+12}{x+2}\cdot\frac{1}{x+4}$$

$$= \lim_{x\to -4} \frac{3\cancel{(x+4)}}{(x+2)\cancel{(x+4)}}$$

$$= \lim_{x\to -4} \frac{3}{x+2}$$

$$\overset{QL}{=} \frac{\lim\limits_{x\to -4} 3}{\lim\limits_{x\to -4}(x+2)}$$

$$= \frac{3}{(-4)+2} = \frac{3}{-2} = -\frac{3}{2}$$

Ex: Calculate $\lim\limits_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{\frac{x-2}{x+2}}$.

Sol: $\lim\limits_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{\frac{x-2}{x+2}}$

$$= \lim_{x\to 2} \left(\frac{1}{x}-\frac{1}{2}\right)\cdot\frac{x+2}{x-2}$$

$$= \lim_{x\to 2} \left(\frac{2}{2}\cdot\frac{1}{x} - \frac{1}{2}\cdot\frac{x}{x}\right)\cdot\frac{x+2}{x-2}$$

$$= \lim_{x\to 2} \left(\frac{2-x}{2x}\cdot\frac{x+2}{x-2}\right)$$

$$= \lim_{x\to 2} \frac{-\cancel{(x-2)}(x+2)}{2x\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{(-1)(x+2)}{2x}$$

$$\overset{QL}{=} \frac{\lim\limits_{x\to 2}\left((-1)(x+2)\right)}{\lim\limits_{x\to 2}(2x)}$$

$$= \frac{(-1)((2)+2)}{2(2)}$$

$$= \frac{(-1)(4)}{4} = -1$$

Ex: Calculate $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$.

Sol: $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} \rightsquigarrow \frac{3-3}{\sqrt{3+1}-2} \rightsquigarrow \frac{0}{0}$ ← Can't apply QL...

$= \lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$ ⟶ "$a^2-b^2=(a+b)(a-b)$ Multiply by algebraic conjugate"

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1})^2+\cancel{2\sqrt{x+1}}-\cancel{2\sqrt{x+1}}-4}$

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{x+1}+2)}{\cancel{x-3}}$

$= \lim_{x\to 3} (\sqrt{x+1}+2)$

$= \sqrt{3+1}+2$

$= \sqrt{4}+2$

$= 2+2 = 4$ $\boxed{\zeta}$

Ex: Calculate $\lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2}$.

Sol: $\lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} \rightsquigarrow \frac{\sqrt{2-1}-1}{2-2} \rightsquigarrow \frac{1-1}{2-2} \rightsquigarrow \frac{0}{0}$

$= \lim_{x\to 2} \left(\frac{\sqrt{x-1}-1}{x-2} \cdot \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} \right)$

$= \lim_{x\to 2} \frac{(x-1)-1}{(x-2)(\sqrt{x-1}+1)}$

$= \lim_{x\to 2} \frac{\cancel{x-2}\ 1}{\cancel{(x-2)}(\sqrt{x-1}+1)}$

$= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} = \frac{1}{\sqrt{2-1}+1} = \frac{1}{1+1} = \frac{1}{2}$ $\boxed{\zeta}$