SL_k-Tilings and Paths in $\mathbb{Z}^k$

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History of Friezes

- Friezes were first studied in the 1970's by Conway and Coxeter in the $k = 2$ case.
- They are in bijection with triangulations of polygons.
- Work was expanded for higher k by Morier-Genoud, Bergeron and Reutenauer in the 2000's.
- In 2020, Bauer, Faber, Gratz, Serhiyenko, and Todorov related friezes to Plücker coordinates.

What is a Frieze?

Definition

An SL_k-*frieze* is an array of offset bi-infinite rows of integers consisting of $k-1$ rows of zeros at the top and bottom, a row of ones below and above them, respectively, rows of integers satisfying the following properties.

- Every $k \times k$ diamond of neighboring entries has determinant 1.
- Every $(k+1) \times (k+1)$ diamond has determinant 0.

Example ($k = 2$)

$$
\begin{array}{cccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 1 & & 3 & & 1 & & 3 & & \cdots \\
 & & 2 & & 2 & & 2 & & 2 & \\
 & 1 & & 1 & & 1 & & 1 & & \\
 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

- If all entries outside the rows of zeros are positive, we call the frieze *positive*.
- If the frieze does not terminate, then we call it an *infinite frieze*.

Plücker Coordinates

- A Plücker coordinate is a map from the Grassmannian to $\mathbb{R}$.
- If an element of the Grassmannian is thought of as a $k \times n$ matrix A, then p_I is the determinant of the submatrix of A whose columns are indexed by I.
- The coordinate ring of the Grassmannian is a cluster algebra and the Plücker coordinates are cluster variables.

Plücker Friezes

- A Plücker coordinate is *consecutive* if its indices are.
- It is *almost consecutive* if it has only one index not in a consecutive set.

Definition

The *Plücker frieze* of type (k, n) denoted by $\mathcal{F}_{(k,n)}$ is an infinite offset array of height $n + k - 1$ whose entries are Plücker coordinates.

Example ($\mathcal{F}_{(3,5)}$)

$$
\begin{array}{cccccccccc}
 & & 0 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & & 0 & & \\
 & & p_{123} & & p_{234} & & p_{345} & & p_{145} & \\
\cdots & p_{135} & & p_{124} & & p_{235} & & p_{134} & & \cdots \\
 & & p_{145} & & p_{125} & & p_{123} & & p_{234} & \\
 & 0 & & 0 & & 0 & & 0 & & \\
 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Theorem (BFGST)

- A Plücker frieze satisfies the properties of a frieze.
- All integer friezes arise by applying the entries of a Plücker frieze to a particular element of the Grassmannian.

Tilings

Definition

A *tiling* $\mathcal{M} = (m_{i\,j})_{i,j\in\mathbb{Z}}$ with values $m_{ij} \in \mathbb{Z}$ is an infinite array. We say that $\mathcal{M}$ is an SL_k*-tiling* if every adjacent $k \times k$ matrix is in $SL_k(\mathbb{Z})$. We say that an SL_k-tiling $\mathcal{M}$ is *tame* if every adjacent $(k+1) \times (k+1)$ sub-matrix of $\mathcal{M}$ has determinant 0. We denote by $\mathbb{SL}_k$ the set of all tame SL_k-tilings

Example (SL_3-tiling)

$$\begin{array}{ccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & 1 & 2 & 3 & 2 & 1 & \cdots \\
\cdots & 1 & 3 & 6 & 4 & 1 & \cdots \\
\cdots & 1 & 2 & 4 & 3 & 1 & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Tilings from Friezes

- Bergeron and Reutanauer devised a method for transforming a frieze F into a tiling $\mathcal{M}_F$.
- Rotate the frieze 45° clockwise and extend periodically.

$$
\begin{array}{ccccccccccc}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
\cdots & 0 & 1 & 2 & 1 & 1 & 0 & \cdots & & & \\
 & \cdots & 0 & 1 & 2 & 3 & 1 & 0 & \cdots & & \\
 & & \cdots & 0 & 1 & 2 & 1 & 1 & 0 & \cdots & \\
 & & & \cdots & 0 & 1 & 2 & 3 & 1 & 0 & \cdots \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
$$

Paths

Definition

Let $\gamma = \{\gamma_i\}_{i \in \mathbb{Z}}$ be a bi-infinite strip of k-column vectors $\gamma_i \in \mathbb{Z}^k$ with the property that the matrix $(\gamma_i, \ldots, \gamma_{i+k-1})$ whose columns are k consecutive entries of γ is an element of $SL_k(\mathbb{Z})$. We denote the set of all such strips $\mathcal{P}_k$ and we call γ a *path*.

- For example,

$$\gamma = \left(\cdots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \cdots \right)$$

- We will frequently refer to $(\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$.
- We mod out by the operation of multiplying each vector in the paths by the same matrix in $SL_k(\mathbb{Z})$.

Motivation

- In 2020, Short gave a bijection between $(\mathcal{P}_2 \times \mathcal{P}_2)/SL_k(\mathbb{Z})$ and $\mathbb{SL}_2$.
- This was done by relating tilings to paths in the Farey Graph.
- He gave several restrictions of this bijection in order to describe all periodic and positive tilings.
- Our goal was to study $\mathbb{SL}_k$ through a similar process.
- The case of $k = 2$ is well-studied in both tilings and friezes, but higher k are not well-understood.

Defining Φ

Definition

Fix a pair $(\gamma, \delta) \in (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$.

$$\begin{aligned} \Phi : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) &\to \mathbb{SL}_k \\ (\gamma, \delta) &\mapsto (m_{i\,j})_{i,j \in \mathbb{Z}}, \end{aligned}$$

where

$$m_{ij} = \det\left(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+k-2}, \delta_j\right).$$

Example $(\Phi(\gamma, \delta))$

$$\gamma = \begin{pmatrix} \cdots & 1 & 0 & 0 & 1 & \cdots \\ & 0 & 1 & 0 & 5 & \\ & 0 & 0 & 1 & 2 & \end{pmatrix}, \quad \delta = \begin{pmatrix} \cdots & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & \\ & 1 & 3 & 6 & \end{pmatrix}$$

$$m_{1\,2} = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 3, \quad \mathcal{M} = \begin{matrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & 1 & {\color{red}3} & 6 & \cdots \\ \cdots & 1 & 1 & 1 & \cdots \\ \cdots & -4 & -3 & -2 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

The Bijection

Theorem (P., Serhiyenko)

The map Φ is a bijection between tame SL_k-tilings and pairs of paths modulo the action by $SL_k(\mathbb{Z})$.

This generalizes the main result of Short for higher dimensions k.

Periodicity

- We say a path γ is *p-periodic* if it has the property that $\gamma_i = \gamma_{i+p}$ for all $i \in \mathbb{Z}$.
- We say that a tiling $\mathcal{M}$ is $(p \times q)$*-periodic* if

$$m_{i\,j} = (-1)^{k-1} m_{i+p\,j} = (-1)^{k-1} m_{i\,j+q} = m_{i+p\,j+q}$$

  for all $i, j \in \mathbb{Z}$.
- This notion was first introduced by Bergeron and Reutenauer.

Corollary

The paths γ and δ are m and n periodic, respectively, if and only if $\mathcal{M} := \Phi(\gamma, \delta)$ is $(m \times n)$-periodic

Dual Tiling

- Bergeron and Reutenauer define the notion of a *dual tiling*.
- The *dual tiling* of $\mathcal{M}$, written $\mathcal{M}^*$ has, as an entry at each position, the $(k-1)$ adjacent minor of the tiling at that entry.
- The dual tiling $\mathcal{M}^*$ is itself a tiling, and $(\mathcal{M}^*)^* = \mathcal{M}$ up to a shift.

Theorem (P., Serhiyenko)

Let $\mathcal{M} = \Phi(\gamma, \delta)$ be an SL_k-tiling. Then its dual $\mathcal{M}^* = \Phi(\delta, \gamma)$.

Tilings from Friezes

- Recall that we can make a tiling $\mathcal{M}_F$ from a frieze by rotating it and repeating it up to shift.
- Define a map ι as follows.

$$\iota : \mathcal{P}_k/SL_k(\mathbb{Z}) \to (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$$
$$(\gamma) \mapsto (\gamma, \gamma).$$

Theorem (P., Serhiyenko)

The restriction of the map Φ given by $\Phi \circ \iota$ is a bijection between tame SL_k-tilings from SL_k-friezes and paths.

Positivity

- Short's results concerning positive friezes and tilings rely heavily on the geometry of the Farey graph.
- We have no connection to from higher Farey graphs to higher dimensional friezes.
- We focus on the study of positive friezes for higher k.

Partial Positivity Results

Lemma

For $k \leq 3$, if the tiling $\mathcal{M}_F$ is positive, then the quiddity sequence is alternating.

- These cases follow from direct applications of our bijection and linear algebra.
- The converse is generally not true.
- We know some cases for higher k values.

Theorem (P.,Serhiyenko)

The tiling $\mathcal{M}_F$ is positive if and only if the quiddity sequence alternates in the following cases.

- $k = 2$ and $n \leq 9$
- $n \leq 8$
  - Except $k = 4$ and $n = 8$
  - $k = 5$ and $n = 8$ has a weird exception

- Cases where $n = k + 2$ are trivial.
- Other cases use results results by Morier-Genoud, Ovsienko, and Tobachnikov.
- The proofs use structures of cluster algebras and the Plücker relations.

Thank You!