Parameterizing semi-simple $U_q(\mathfrak{g})-$module algebras

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Hopf Actions

Definition (Hopf Action)

For an associative $\mathbb{k}-$algebra A, and H a Hopf algebra, we say an action $H \otimes A \to A$ is a Hopf action if

$$h \cdot (xy) = \sum (h_1 \cdot x)(h_2 \cdot y) \text{ for all } x, y \in A \text{ and } h \in H$$

where $\Delta(h) = \sum h_1 \otimes h_2$. We also say A is an $H-$module algebra.

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A Hopf action of $\mathbb{k}G$ on A implies each $g \in G$ acts by automorphism.

$$\Delta(g) = g \otimes g \quad \Rightarrow \quad g \cdot (xy) = (g \cdot x)(g \cdot y)$$

A Hopf action of $U(\mathfrak{g})$ on A implies each $x \in \mathfrak{g}$ acts by derivation.

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \Rightarrow \quad x \cdot (ab) = a(x \cdot b) + (x \cdot a)b$$

Equivalently, the multiplication $A \otimes A \to A$ is an $H-$module morphism.

Quantized Enveloping Algebras

Let $\mathfrak{g}$ be a simple Lie algebra of rank r and $A = (a_{ij})$ its Cartan matrix.

There exists unique relatively prime positive integers $\{d_i\}_{i=1}^r$ such that $d_i a_{ij} = d_j a_{ji}$ for all i, j. Fix $q \in \Bbbk^\times$ and define $q_i = q^{d_i}$.

DEFINITION 5.7.1. The *quantum group* $U_q(\mathfrak{g})$ is generated by elements E_i, F_i and invertible elements K_i, $i = 1, ..., r$, with defining relations

(5.12) $$K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

(5.13) $$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

and the *q-Serre relations*:

(5.14) $$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{q_i}![1 - a_{ij} - l]_{q_i}!} E_i^{1-a_{ij}-l} E_j E_i^l = 0, i \neq j$$

and

(5.15) $$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{q_i}![1 - a_{ij} - l]_{q_i}!} F_i^{1-a_{ij}-l} F_j F_i^l = 0, i \neq j.$$

Quantized Enveloping Algebras (Cont)

The coproduct below defines a unique Hopf algebra structure on $U_q(\mathfrak{g})$:

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i$$
$$\Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1$$
$$\Delta(K_i) = K_i \otimes K_i$$

Question

What data do we need to define an $H-$module algebra structure on A when A is one of the following?

- $A = \mathbb{k}^n$;
- $A = M_n(\mathbb{k})$;
- $A = \prod_{i=1}^{r} M_{n_\lambda}(\mathbb{k})$?

Actions of $U_q(\mathfrak{sl}_2)$

Theorem (Kinser, Oswald (2021))

The following data determines a Hopf action of $U_q(\mathfrak{sl}_2)$ on $\Bbbk^n$.

1. *A permutation action of K;*
2. *A collection of scalars $\gamma_i^E, \gamma_i^F \in \Bbbk$ satisfying:*

$$\gamma_{K\cdot i}^E = q^{-2}\gamma_i^E, \quad \gamma_{K\cdot i}^F = q^2\gamma_i^F \quad \forall i \in \{1, 2, \ldots, n\}$$

and

$$\gamma_i^E\gamma_i^F = \frac{-q}{(1-q^2)^2} \text{ for all } i \text{ satisfying } K^2 \cdot i \neq i$$

The action is given by

$$E \cdot e_i = \gamma_i^E e_i - \gamma_{K\cdot i}^E e_{K\cdot i},$$
$$F \cdot e_i = q^2(\gamma_{K^{-1}\cdot i}^F e_{K^{-1}\cdot i} - \gamma_i^F e_i)$$

Moreover, every Hopf action of $U_q(\mathfrak{sl}_2)$ on $\Bbbk^n$ is of this form.

Actions of $U_q(\mathfrak{g})$ on $\mathbb{k}^n$

The subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i\}$ is isomorphic (as a Hopf algebra) to $U_{q_i}(\mathfrak{sl}_2)$.

The previous proposition implies:

$$E_i \cdot e_s = \gamma_s^{E_i} e_s - \gamma_{K_i \cdot s}^{E_i} e_{K_i \cdot s},$$
$$F_i \cdot e_s = q^2(\gamma_{K_i^{-1} \cdot s}^{F_i} e_{K_i^{-1} \cdot s} - \gamma_s^{F_i} e_s),$$

for some choice of $\gamma_s^{E_i}, \gamma_s^{F_i} \in k$ satisfying $\gamma_s^{E_i} = q_i^2 \gamma_{K_i \cdot s}^{E_i}$ and $\gamma_s^{F_i} = q_i^{-2} \gamma_{K_i \cdot s}^{F_i}$.

The additional relations will impose more conditions on these scalars.

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for some choice of $\gamma_s^{E_i}, \gamma_s^{F_i} \in k$ satisfying $\gamma_s^{E_i} = q_i^2 \gamma_{K_i \cdot s}^{E_i}$ and $\gamma_s^{F_i} = q_i^{-2} \gamma_{K_i \cdot s}^{F_i}$.

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Proposition

Suppose $\mathfrak{g}$ is a simple Lie algebra and $Rank(\mathfrak{g}) > 1$. Then every Hopf action of $U_q(\mathfrak{g})$ on $\mathbb{k}^n$ factors through a group algebra:

$$U_q(\mathfrak{g}) / \langle E_i, F_i, K_i^2 - 1 \rangle \cong \mathbb{k}[(\mathbb{Z}/2\mathbb{Z})^{Rank(\mathfrak{g})}].$$

Actions of $U_q(\mathfrak{sl}_2)$ on $M_n(\Bbbk)$

Recall, $U_q(\mathfrak{sl}_2)$ is generated by E, F, K and:

$$\begin{aligned}\Delta(E) &= 1 \otimes E + E \otimes K \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1 \\ \Delta(K) &= K \otimes K\end{aligned}$$

Proposition

For every $\varphi \in Aut\,(M_n(\Bbbk))$, there exists $G \in GL_n(\Bbbk)$ such that $\varphi(X) = G^{-1}XG$ for all $X \in M_n(\Bbbk)$.

Proposition

For every $f \in Der\,(M_n(\Bbbk))$, there exists $H \in M_n(\Bbbk)$ such that $f(X) = [H, X]$ for all $X \in M_n(\Bbbk)$.

Actions of $U_q(\mathfrak{sl}_2)$ on Matrix Algebras

Proposition

The following data defines a Hopf action of $U_q(\mathfrak{sl}_2)$ on $M_n(\Bbbk)$.

1. *An invertible matrix* $\mathtt{K}$,
2. *A matrix* $\mathtt{E}$ *satisfying* $\mathtt{KE} = q^{-2}\mathtt{EK}$,
3. *A matrix* $\mathtt{F}$ *satisfying* $\mathtt{KF} = q^{2}\mathtt{FK}$,

and additionally satisfying

$$\mathtt{FE} - q^2\mathtt{EF} - \frac{\mathtt{K}^{-2}}{q - q^{-1}} \in Z(M_n(\Bbbk)).$$

The action is then given by:

$$K \cdot \mathtt{M} = \mathtt{K}^{-1}\mathtt{MK}, \qquad E \cdot \mathtt{M} = [\mathtt{E}, \mathtt{M}]\mathtt{K}, \qquad F \cdot \mathtt{M} = \mathtt{K}[\mathtt{F}, \mathtt{M}],$$

for all $\mathtt{M} \in M_n(\Bbbk)$ *and every such Hopf action is of this form.*

Actions of $U_q(\mathfrak{g})$ on Matrix Algebras

Again, the restriction to subalgebras $\langle K_i, E_i, F_i \rangle$ implies:

$$K_i \cdot \mathtt{M} = \mathtt{K}_i^{-1}\mathtt{M}\mathtt{K}_i$$
$$E_i \cdot \mathtt{M} = [\mathtt{E}_i, \mathtt{M}]\mathtt{K}_i$$
$$F_i \cdot \mathtt{M} = \mathtt{K}_i[\mathtt{F}_i, \mathtt{M}]$$

for a collection of $\mathtt{K}_i, \mathtt{E}_i, \mathtt{F}_i \in M_n(\Bbbk)$. Additionally, we require

$$\mathtt{K}_i\mathtt{E}_i = q_i^{-2}\mathtt{E}_i\mathtt{K}_i,$$
$$\mathtt{K}_i\mathtt{E}_i = q_i^{2}\mathtt{F}_i\mathtt{K}_i$$
$$\mathtt{F}_i\mathtt{E}_i - q_i^2\mathtt{E}_i\mathtt{F}_i - \frac{\mathtt{K}_i^{-2}}{q_i - q_i^{-1}} \in Z(M_n(\Bbbk))$$

We have more relations to satisfy.

Relations

Algebra Relations	Matrix Relations
$K_i K_j = K_j K_i$	$\mathtt{K}_i \mathtt{K}_j = c_{ij} \mathtt{K}_i \mathtt{K}_j$
$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j$	$\mathtt{K}_i \mathtt{E}_j = c_{ij}^{-1} q_i^{-a_{ij}} \mathtt{E}_j \mathtt{K}_i$
$K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$	$\mathtt{K}_i \mathtt{F}_j = c_{ij}^{-1} q_i^{a_{ij}} \mathtt{F}_j \mathtt{K}_i$
$[E_i, F_j] = 0$	$\mathtt{F}_j \mathtt{E}_i - q_i^{a_{ij}} \mathtt{E}_i \mathtt{F}_j \in Z(M_n(\Bbbk))$

Relations

Lastly, we have the q-Serre relations:

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \binom{1-a_{ij}}{\ell}_{q_i} E_i^{1-a_{ij}-\ell} E_j E_i^\ell = 0$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \binom{1-a_{ij}}{\ell}_{q_i} F_i^{1-a_{ij}-\ell} F_j F_i^\ell = 0$$

which implies

$$\sum_{\ell=0}^{1-a_{ij}} (-c_{ij})^\ell \binom{1-a_{ij}}{\ell}_{q_i} \mathrm{E}_i^{1-a_{ij}-\ell} \mathrm{E}_j \mathrm{E}_i^\ell \in Z(M_n(\Bbbk))$$

$$\sum_{\ell=0}^{1-a_{ij}} (-c_{ij})^\ell \binom{1-a_{ij}}{\ell}_{q_i} \mathrm{F}_i^{1-a_{ij}-\ell} \mathrm{F}_j \mathrm{F}_i^\ell \in Z(M_n(\Bbbk))$$

This data defines a Hopf action of $U_q(\mathfrak{g})$ on $M_n(\Bbbk)$.

Actions of $U_q(\mathfrak{sl}_2)$ on Products of Matrix Algebras

Question

What are the automorphisms and skew-derivations of $A := \prod_{r=1}^{n} M_{\lambda_r}(\mathbb{k})$?

1. Every automorphism is the composition of an inner automorphism and a permutation of factors of the same dimension.
2. If f_σ is the automorphism induced by the permutation σ, then a $(1, f_\sigma)$-skew derivation D is determined by some $X \in \prod_{r=1}^{n} M_{\lambda_r}(\mathbb{k})$ and is defined by:

$$D(Y) = [X, Y]$$

Skew Derivations

If f_σ permutes some factor $M_{\lambda_r}(\mathbb{k})$ then every $(1, f_\sigma)$ skew derivation D, restricts to the zero map on $M_{\lambda_r}(\mathbb{k})$.

Actions of $U_q(\mathfrak{sl}_2)$ on Products of Matrix Algebras (cont.)

Note

For any factors permuted by K, both E and F must act by zero.

Proposition

A Hopf action of $U_q(\mathfrak{sl}_2)$ on $A := \prod_{r=1}^{n} M_\lambda(\Bbbk)$ with transitive (on the factors) K action either:

1. *has 2 factors and the action factors through through the group algebra $\Bbbk[\mathbb{Z}/2\mathbb{Z}]$, or*
2. *has 1 factor and falls under the previous classification for matrix algebras.*

Finding an appropriate form of equivalence

When does different data define the "same" $U_q(\mathfrak{g})-$module algebra?

$A-$Modules

Given a $U_q(\mathfrak{g})-$module algebra A. We say $M \in \mathrm{Rep}(U_q(\mathfrak{g}))$ is a left $A-$module if there is a $U_q(\mathfrak{g})-$module morphism $\rho : A \otimes M \to M$ such that the following maps are equal:

$$\rho \circ (1 \otimes \rho) : A \otimes A \otimes M \to M$$
$$\rho \circ (m \otimes 1) : A \otimes A \otimes M \to M$$

Similarly, we can define right $A-$modules and $(A, A')-$bimodules. And these constructions form abelian subcategories of $\mathrm{Rep}(U_q(\mathfrak{g}))$.

We can talk about equivalence of module categories.

Bimodule Categories

Let S_m denote the $U_q(\mathfrak{sl}_2)-$module algebra with $S_m \cong \Bbbk^m$ as $k-$algebras.

Kinser-Oswald (2021)

The category $(S_m, S_{m'})-$Bimod is equivalent to Rep(Q, J) for a quiver Q with relations J and Q depends only on q, m, m'.

Note: The path algebra $\Bbbk Q/J$ has finite-dimensional quotients of wild representation type.

Since the characterization of $U_q(\mathfrak{g})-$module algebras of the form $\Bbbk^n$ are more simple in the case where Rank$(\mathfrak{g}) > 1$, a similar result should hold for $U_q(\mathfrak{g})-$module algebras.

Directions of Further Study

1. Finish the classification of Hopf actions of $U_q(\mathfrak{g})$ on $\prod_{i=1}^r M_n(\mathbb{k})$.
2. Investigate the module categories for the non-commutative algebras discussed.
3. Investigate the actions which factor through the small quantum groups, $u_q(\mathfrak{g}) = \frac{U_q(\mathfrak{g})}{\langle E_i^\ell, F_i^\ell, K_i^\ell - 1\rangle}$ where q is an ℓ^{th} root of unity, $\ell \geq 3$ odd.

Thank You/Questions

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