Differentials on $\mathbb{F}_1$

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Monoids with Real Degree

$$\begin{aligned} \mathbb{F}_1[X_1, \ldots, X_n] &:= \{0\} \cup \{X_1^{u_1} \cdots X_n^{u_n} \text{ such that } u_i \in \mathbb{R}_{\geqslant 0}\}, \\ \mathbb{F}_1(X_1, \ldots, X_n) &:= \{0\} \cup \{X_1^{u_1} \cdots X_n^{u_n} \text{ such that } u_i \in \mathbb{R}\}. \end{aligned} \tag{1}$$

The first line defines the polynomial rings and the second line is the field of fractions. $\mathbb{R}$ can be replaced by $\mathbb{Z}$ or $\mathbb{Q}$.

Category

Let us now construct a category on the monomials $A := \mathbb{F}_1(X_1, \ldots, X_n)$.
Objects as Sets Fix $\varepsilon \in \mathbb{R}$, then define the sets by valuations,

$$A(\varepsilon, X_i) := \{m \in A \text{ such that } \mathrm{val}_{X_i}(m) \geqslant \varepsilon \text{ for some given } \varepsilon \in \mathbb{R}\}.$$

Every set contains 0 since $\mathrm{val}(0) = \infty$.
Example 1. $A(2, X_1) = \{X_1^2, X_1^2 X_2^{-1}, X_1^3 X_3, \ldots\}$.
2. $A(-2, X_1) = \{X_2, X_1^{-2}, X_1^{-1} X_2^2, X_1^3 X_3, \ldots\}$.
Ideals The sets $A(\varepsilon, X_i)$ are modeled after ideals for $\varepsilon \geqslant 0$. For example, in the integer degree case the ideal generated by $\langle X \rangle$ is $A(1, X)$. These sets are multiplicatively closed. We can also define prime sets which become analogues of prime ideals.

Morphisms as inclusion maps The family of sets are ordered by inclusion as in a topological space.

$$\mathrm{Mor}(U,V) = \begin{cases} \{\mathrm{pt}\} & \text{if } U \subseteq V \\ \varnothing & \text{otherwise,} \end{cases} \tag{2}$$

where $\{\mathrm{pt}\}$ is a set with single element.

Examples Let $\delta > 0$ then $A(\varepsilon, X_i) \subset A(\varepsilon - \delta, X_i)$ for any given $\varepsilon \in \mathbb{R}$. For example,

$$\cdots \subset A(2, X_0) \subset A(1, X_0) \subset A(0, X_0) \subset A(-1, X_0) \subset \cdots .$$

Morphisms as Partial Derivatives The idea of set inclusion as a morphism can be carried onto the setting of partial derivatives by constructing set inclusions of the form

$$A(\varepsilon, X_i) \subset A(\varepsilon - 1, X_i).$$

Let $a \in A$ be a monomial and $\partial_i a$ denote the partial derivative of a with respect to X_i, that is $\partial a / \partial X_i$. There are two possibilities for a either $\mathrm{val}_{X_i}(a) \neq 0$ or $\mathrm{val}_{X_i}(a) = 0$.

$$\partial_i a = \begin{cases} a/X_i & \text{if } \mathrm{val}_{X_i}(a) \neq 0, \\ a & \text{otherwise.} \end{cases} \tag{3}$$

Thus there are inclusion morphisms of the form

$$A(\varepsilon, X_i) \xrightarrow{\partial_i} A(\varepsilon - 1, X_i). \tag{4}$$

Example $\partial_X(X^2Y) = XY$ and $\partial_Z(X^2Y) = X^2Y$.

Antiderivatives

The antiderivatives are straightforward as well. If a is a monomial then

$$\int_{X_i} a = a \cdot X_i.$$

The above operation is not defined for $a = 1/X_i$.

Base Category

Let $\mathcal{B}$ denote the category with objects as sets $A(\varepsilon, X_i)$. The morphisms are identity maps for each object and partial derivatives. This category will be referred to as the base category.

Fibered Category In order to keep track of coefficients, which come from the exponents of monomials after differentiation, a fibered category needs to be constructed over a the base category $\mathcal{B}$. This new category will be denoted by $\mathcal{C}$.
Objects of the Category The category $\mathcal{C}$ is constructed so that its objects behave as vector bundles over the base category $\mathcal{B}$. An object of category $\mathcal{C}$ is indexed by elements of an object of the base category $\mathcal{B}$. The object is of the form

$$\prod_{a \in A(\varepsilon, X_i)} K_a \tag{5}$$

where $K_a = (c, a)$ is a pair with $c \in K, a \in A(\varepsilon, X_i)$. K is a ring or a field which keeps track of coefficients after differentiation.

Morphisms as Partial Derivatives The morphisms in this category are constructed over the morphisms of the base category, that is, $A(\varepsilon, X_i) \xrightarrow{\partial_1} A(\varepsilon - 1, X_i)$.

$$\begin{aligned} \prod_{a \in A(\varepsilon, X_i)} K_a \xrightarrow{\partial_i} \prod_{a' \in A(\varepsilon-1, X_i)} K_{a'} \\ \partial_i(c, a) = \begin{cases} (0, a) & \text{if } \nu_{X_i}(a) = 0 \\ (c \cdot \nu_{X_i}(a), a/X_i), & \text{otherwise} \end{cases} \end{aligned} \tag{6}$$

Example

1. $\frac{\partial}{\partial X}(2, X^2Y) = (4, XY)$ where $X^2Y \in A(2, X)$ and $XY \in A(1, X)$.
2. $\frac{\partial}{\partial Z}(2, X^2Y) = (0, X^2Y)$ where $X^2Y \in A(2, X) \subset A(1, X)$.

Polynomials as sections

A polynomial with coefficients in K can be expressed as a section of $\prod K_a$. For example, $1 + 2X + X^2 + 3Y$ is a section of $\prod_{a \in A(0,X)} \mathbb{Z}_a$ given as a tuple

$$(1, 1), (2, X), (1, X^2), (3, Y), (0, X^{10}), \ldots$$

with infinitely many zero coefficients. The partial derivative with respect to X will give

$$(0, 1), (2, 1), (2, X), (0, Y), (0, X^9), \ldots$$

which translates to the polynomial $2 + 2X$.

Series as sections

The series for sine and cosine can be considered as sections of $\prod_{a \in A(0,X)} \mathbb{Q}_a$. For $\sin X$ the sections are

$$(1, X), (-1/3!, X^3), (1/5!, X^5), (-1/7!, X^7), \ldots, ((-1)^n/(2n+1)!, X^{2n+1}), \ldots$$

and $(0, m)$ for all other monomials. Taking the derivative gives cosine

$$(1, 1), (-1/2!, X^2), (1/4!, X^4), (-1/6!, X^6), \ldots, ((-1)^n/(2n)!, X^{2n}), \ldots$$

Antiderivatives

The antiderivatives in a fibered category require a bit more care, since integration introduces denominators which may not have inverses in the given ring or a field.

Let (c, a) be the pair where $c \in K$ and a is a monomial (not equal to $1/X_i$) then

$$\int_{X_i} (c, a) = \left(\frac{c}{\mathrm{val}_{X_i}(a) + 1}, a \cdot X_i \right) \tag{7}$$

Char p A major problem while working over the field $\mathbb{F}_p$ is the following integral

$$\int_X X^{p-1} = \frac{X^p}{p}, \tag{8}$$

where the right hand side is not defined. This problem does not occur over $\mathbb{F}_1$ since the coefficient $1/p$ is ignored. But, the field in the fibered category has to be chosen carefully.

Total Derivative

The total derivative merges the partial derivatives given by one forms. The same constructions work as in the previous section.

One Forms Ω^1

The one forms consists of terms like $f dX_i$ where $f \in A(\varepsilon, X_i)$. This can be fit into a base category and then total derivative can be defined as a section of an object in the fibered category.

Base category

The base category will be denoted by $\mathcal{B}'$.

Objects of Base category

The objects are sets of the form $\cap_i A(\varepsilon_i, X_i)$ and $\cap_i A(\varepsilon_i, X_i)dX_i$. In other words each X_i has valuation at least ε_i.

Morphisms of Base category

Every object has an identity map. The only morphisms that exist between distinct objects come from applying ∂_i to the monomial for each i.

$$\begin{aligned} \cup_j \partial_j : \cap A(\varepsilon_i, X_i) &\to \cap A(\varepsilon_i - 1, X_i)dX_i \\ f &\mapsto \cup_j (\partial_j f)dX_j \end{aligned} \tag{9}$$

Fibered category

The fibered category will be denoted by $\mathcal{C}'$.

Objects of fibered category

The objects are direct products indexed by the base set

$$\prod_{a \in \cap A(\varepsilon, X_i)} K_a \quad \text{and} \quad \prod_{a \in \cap A(\varepsilon, X_i) d X_i} K_a.$$

Morphisms of fibered category Every object has an identity map. The only morphisms that exist between distinct objects come from $\cup_i \partial_i$.

Example

$$
\begin{aligned}
(\cup\partial)(c, X^2YZ) &\mapsto (2c, XYZdX), (c, X^2ZdY), (c, X^2YdZ) \\
(\cup\partial)(X^2YZ) &\mapsto (XYZdX) \cup (X^2ZdY) \cup (X^2YdZ)
\end{aligned} \tag{10}
$$

The total derivative in the fibered category is the analogue of

$$2c \cdot XYZdX + c \cdot X^2ZdY + c \cdot X^2YdZ.$$

Arithmetic Derivative

The above example recreates the notion of arithmetic derivative as given in wikipedia. Setting $c = 1, X = 2, Y = 3, Z = 5$ and dropping dX, dY, dZ gives total derivative in fibered category as

$$(2, 30), (1, 20), (1, 12).$$

Summing the above as $2 \cdot 30 + 1 \cdot 20 + 1 \cdot 12 = 92$ which is arithmetic derivative of 60.

Leibniz Rule

The fibered category allows us to confirm Leibniz rule for partial derivatives. Let a, b be monomials such that $\nu_{X_i}(a)$ and $\nu_{X_i}(b) > 0$ then

$$\partial_i(ab) = a\partial_i b = b\partial_i a = \frac{ab}{X_i}. \tag{11}$$

The coefficients can be allowed in the fibered category.

$$\begin{aligned}(c, ab) &\xrightarrow{\partial_i} (c \cdot \nu_{X_i}(b), a\partial_i b) \oplus (c \cdot \nu_{X_i}(a), b\partial_i a) \\ &= (c \cdot \nu_{X_i}(b) + c \cdot \nu_{X_i}(a))a\partial_i b \\ &= (c \cdot \nu_{X_i}(b) + c \cdot \nu_{X_i}(a))b\partial_i a.\end{aligned} \tag{12}$$

Example

The partial derivative ∂ of cX^5 according to Leibniz rule and in fibered category is given below.

$$\begin{aligned} \partial(cX^3 \cdot X^2) &= cX^3\partial X^2 + cX^2\partial X^3 \\ &= (2c+3c)X^4 = 5cX^4, \\ cX^3 \cdot X^2 &\xrightarrow{\partial} (c, X^3\partial X^2) \oplus (c, X^2\partial X^3) \\ &= (2c, X^4) \oplus (3c, X^4) = (5c, X^4). \end{aligned} \tag{13}$$

In the base category the Leibniz rule is the equality

$$\partial(X^3 \cdot X^2) = X^3\partial X^2 = X^2\partial X^3 = X^4 = \frac{X^5}{X}.$$

Example

What if the valuation is zero? Below $\partial := \partial_X$.

$$\begin{aligned}\partial(cX^3 \cdot Y) &= cX^3\partial Y + cY\partial X^3 \\ &= 0 + 3cX^2Y = 3cX^2Y, \\ cX^3 \cdot Y &\xrightarrow{\partial} (c, X^3\partial Y) \oplus (c, Y\partial X^3) \\ &= (0, X^3Y) \oplus (3c, X^2Y).\end{aligned} \tag{14}$$

Since there is no addition in the Base category there is no interpretation of Leibniz rule.

de Rham Cohomology

In this section we will use integer degrees in $\mathbb{F}_1[X_1, \ldots, X_n]$.

The cohomology occurs in the fibered category, which keeps track of the coefficients and the negative signs coming from the alternating algebra, that is $dX_i dX_j = -dX_j dX_i$. The base category keeps track of the negative sign by making the string $dX_i dX_j$ non commutative.

Zero Forms

The zero forms is a set of all monomials.

$$\Omega^0 := \mathbb{F}_1[X_1, \ldots, X_n]. \tag{15}$$

One forms

These are defined as

$$\Omega^1 := \{f dX_i \text{ such that } f \in \mathbb{F}_1[X_1, \ldots, X_n]\}. \tag{16}$$

2 Forms

The two forms have non commutative differentials, that is $dX_i dX_j \neq dX_j dX_i$.

$$\begin{aligned} \Omega^1 :=& \{f dX_i dX_j \text{ such that } f \in \mathbb{F}_1[X_1, \ldots, X_n] \\ & i \neq j \text{ and } dX_i dX_j \text{ non commutative}\}. \end{aligned} \tag{17}$$

k forms

The string of differentials is non commutative. Let k elements be chosen from $\{1, 2, \ldots, n\}$ and denoted as $\{i_1, \ldots, i_k\}$, arranged in increasing order. If σ, τ are two distinct permutations of the k element set, then

$$dX_{\sigma(i_1)} \ldots dX_{\sigma(i_k)} \neq dX_{\tau(i_1)} \ldots dX_{\tau(i_k)}.$$

The k forms are defined as

$$\begin{aligned}\Omega^k := &\{f dX_{\sigma(i_1)} \ldots dX_{\sigma(i_4 k)} \text{ such that } f \in \mathbb{F}_1[X_1, \ldots, X_n] \\ &\sigma(i_j) \text{ distinct and } dX_{\sigma(i_1)} \ldots dX_{\sigma(i_k)} \text{ non commutative}\}.\end{aligned} \tag{18}$$

Base Category

The base category consists of $n+1$ objects, each object a set $\Omega^i, 0 \leqslant i \leqslant n$. The morphisms are given as $\cup_i \partial_i$

$$\begin{gathered}\cup_i \partial_i : \Omega^{k-1} \to \Omega^k \\ \cup_i \partial_i : f dX_{\sigma(i_1)} \dots dX_{\sigma(i_{k-1})} \mapsto \cup_i (\partial_i f) dX_{\sigma(i_1)} \dots dX_{\sigma(i_{k-1})} dX_i,\end{gathered} \tag{19}$$

such that $dX_{\sigma(i_1)} \dots dX_{\sigma(i_{k-1})} dX_i = 0$ if there is a repetition of dX_i.

Example

Let $\Omega^0 = \mathbb{F}_1[X_1, X_2, X_3, X_4]$ and $X_1^2 X_2^3 dX_1 dX_3 \in \Omega^2$, then the morphism $\Omega^2 \to \Omega^3$ is given as

$$\begin{aligned}(\cup_{i=1}^4 \partial_i) X_1^2 X_2^3 dX_1 dX_3 &= \partial_1 X_1^2 X_2^3 dX_1 dX_3 \cup \partial_2 X_1^2 X_2^3 dX_1 dX_3 \\ &\cup \partial_3 X_1^2 X_2^3 dX_1 dX_3 \cup \partial_4 X_1^2 X_2^3 dX_1 dX_3 \\ &= 0 \cup X_1^2 X_2^2 dX_1 dX_3 dX_2 \cup 0 \cup X_1^2 X_2^3 dX_1 dX_3 dX_4 \in \Omega^3.\end{aligned} \tag{20}$$

Fibered Category

Define $\mathrm{sgn}(\sigma) = (-1)^{N(\sigma)}$ where $N(\sigma)$ is the number of inversions required to arrange the k elements to increasing order. There are n objects of the form

$$\prod_{a \in \Omega^i} K_a \text{ such } 0 \leqslant i \leqslant n$$

and K_a is a tuple of the form $(\mathrm{sgn}(\sigma) \cdot c, a)$ with $c \in K, a \in \Omega^i$. The only difference is that the string $dX_{\sigma(i_1)} \dots dX_{\sigma(i_k)}$ is now commutative with sign of permutation absorbed in the coefficient. The morphisms are given as a union of partials

$$\cup_i \, \partial_i : \prod_{a \in \Omega^{k-1}} K_a \to \prod_{a \in \Omega^k} K_a. \tag{21}$$

Example

1. Let $\Omega^0 = \mathbb{F}_1[X_1, X_2, X_3, X_4]$ and $(c, X_1^2 X_2^3 dX_1 dX_3) \in \Omega^2$, then the morphism $\Omega^2 \to \Omega^3$ is given as

$$
\begin{aligned}
&(\cup_{i=1}^4 \partial_i)(c, X_1^2 X_2^3 dX_1 dX_3) \\
&= \partial_1(c, X_1^2 X_2^3 dX_1 dX_3), \partial_2(c, X_1^2 X_2^3 dX_1 dX_3), \partial_3(c, X_1^2 X_2^3 dX_1 dX_3), \partial_4(c, X_1^2 X_2^3 dX_1 dX_3) \\
&= 0, (3c, X_1^2 X_2^2 dX_1 dX_3 dX_2), 0, (0, X_1^2 X_2^3 dX_1 dX_3 dX_4) \in \Omega^3 \\
&= 0, (-3c, X_1^2 X_2^2 dX_1 dX_2 dX_3), 0, (0, X_1^2 X_2^3 dX_1 dX_3 dX_4) \in \Omega^3
\end{aligned} \tag{22}
$$

2. $d^2 = 0$

$$(c, X^2 Y) \to (2c, XY dX), (c, X^2 dY) \to (2c, X dX dY), (2c, X dY dX).$$

But $(c, f dY dX) = (-c, f dX dY)$, thus summing up over $f dX dY$ gives $2c - 2c = 0$.

Complex

Proposition

The fibered category leads to a complex

$$\prod 0 \to \prod_{a \in \Omega^1} K_a \to \prod_{a \in \Omega^2} K_a \to \dots. \tag{23}$$

with map $d = \cup_i \partial_i$ and $d^2 = 0$.

If $-1 \in \mathbb{F}_1[X_1, \ldots, X_n]$ such that $-1 + 1 \equiv 0$, the de Rham cohomology can be replicated in the base category itself by setting

$$dX_{\sigma(i_1)} \cdots dX_{\sigma(i_k)} = \operatorname{sgn}(\sigma) dX_{i_1} \cdots dX_{i_k}.$$

Proposition

The following is a complex

$$0 \to \Omega^1 \to \Omega^2 \to \ldots \tag{24}$$

with map $d = \cup_i \partial_i$ and $d^2 = 0$.

Proposition

Let d^k be the differential $\Omega^k \to \Omega^{k+1}$ for $k \geqslant 1$. Then
1. $\operatorname{Ker} d^1 = \mathbb{F}_1[X_1, \ldots, X_n]$.
2. $\operatorname{Ker} d^{k+1} = \operatorname{Im} d^k$.

Let $\mathcal{D}$ denote the base category.

Objects

Let S be a subset of $\{1, 2, \ldots, n\}$. The objects of $\mathcal{D}$ are sets denoted by U_S and defined as

$$U_S = \cap_s B(0, X_s) \cup \{1\} \text{ where } i \in S. \tag{25}$$

If S is a singleton then $U_i := B(0, X_i) \cup \{1\}$. For more than one element in S there are intersections $U_{i,j} = U_i \cap U_j = B(0, X_i) \cap B(0, X_j) \cup \{1\}$. If S is empty then set $U_\varnothing := \mathbb{F}_1[X_1, \ldots, X_n]$, on the other hand if S is the entire set then $U_S = \cup_{u>0}\{X_1^{u_1} \cdots X_n^{u_n}\} \cup \{1\}$ where u is the vector $(u_1, \ldots, u_n)$ and each $u_i > 0$. In other words every monomial in U_S contains all the variables.

Morphisms

The morphisms are inclusions Maps. If $S_i \subseteq S$ and $S_1 \subseteq S_2$, then $U_{S_1} \hookleftarrow U_{S_2}$. For example

$$U_i \hookleftarrow U_{i,j} \hookleftarrow U_{i,j,k}.$$

Fibre Products and Limits

Since, the morphisms are simple inclusion maps

Fibre Product	Intersection
Direct Limit	$U_\varnothing$
Inverse Limit	U_S

Opposite category

Let $\mathcal{E}$ denote the category which serves as an opposite category to $\mathcal{D}$. In fact $\mathcal{E}$ is constructed so that it is isomorphic to $\mathcal{D}^o$ but with different objects.

Objects

Let S be a subset of $\{1, 2, \ldots, n\}$. The objects of $\mathcal{E}$ are sets denoted by V_S and defined as

$$\begin{aligned} V_S := &\{m \in \mathbb{F}_1(X_1, \ldots, X_n) \text{ such that } \mathrm{val}_{X_s}(m) \in \mathbb{R} \text{ for } s \in S \\ &\text{and } \mathrm{val}_{X_j}(m) > 0 \text{ for } j \in \{1, 2, \ldots, n\} \backslash S\}. \end{aligned} \tag{26}$$

For example if S is a singleton then

$$V_i := \{m \in \mathbb{F}_1(X_1, \ldots, X_n) \text{ such that } \mathrm{val}_{X_i}(m) \in \mathbb{R} \text{ and } \mathrm{val}_{X_j}(m) > 0 \text{ for } i \neq j\}. \tag{27}$$

Examples

1. The elements of the set V_X in $\mathbb{F}_1(X, Y, Z)$ are $\{1, X, Y, Y/X, Y^2/X^3, Y^2Z^2/X, \ldots\}$. This is an analogue of localization of $\mathbb{F}_1[X, Y, Z]$ at the multiplicatively closed set $\{1, X, X^2, \ldots\}$, that is, $\mathbb{F}_1[X, Y, Z][1/X]$.

2. The elements of the set $V_X \cap V_Y$ in $\mathbb{F}_1(X, Y, Z)$ is $\mathbb{F}_1[X, Y, Z]$.

3. The elements of the set V_{XY} are $\{1, X, Y, Y/X, Y^2/X^3, 1/Y^2, Z/XY \ldots\}$. This is an analogue of localization of $\mathbb{F}_1[X, Y, Z][1/X]$ at the multiplicatively closed set $\{1, Y, Y^2, \ldots\}$, that is, $\mathbb{F}_1[X, Y, Z][1/X, 1/Y]$. Since $\{X\} \subset \{X, Y\}$ therefore $V_X \subset V_{XY}$.

4. If S is the empty set then the definition immediately gives $V_\varnothing = \mathbb{F}_1[X_1, \ldots, X_n]$, on the other hand if S is the entire set, then the definition gives $V_S = \mathbb{F}_1(X_1, \ldots, X_n)$.

Morphisms

The morphisms are inclusion maps. If $S_i \subseteq S$ and $S_1 \subseteq S_2$, then $V_{S_1} \hookrightarrow V_{S_2}$. For example

$$V_i \hookrightarrow V_{i,j} \hookrightarrow V_{i,j,k}.$$

Fibre Products and Limits

Since, the morphisms are simple inclusion maps

Fibre Product	Intersection
Direct Limit	V_S
Inverse Limit	$V_\varnothing$

Proposition

Let S be a non empty subset of $\{1, 2, \ldots, n\}$ and $S_i \subseteq S_j \subseteq S$. Let F be a functor $F : \mathcal{D} \to \mathcal{E}$ which sends an object U_S in category $\mathcal{D}$ to an object V_S in category $\mathcal{E}$. It sends the morphism $U_{S_i} \to U_{S_j}$ in category $\mathcal{D}$ to morphism $V_{S_i} \leftarrow V_{S_j}$ in category $\mathcal{E}$. Then F is a sheaf.

$$
\begin{tikzcd}
 & U_0 \arrow[ddl] & U_0 \cap U_1 \arrow[l] \arrow[ddl] & \\
 & \oplus & \oplus & \\
\mathbb{F}_1[X_0, X_1, X_2] & U_1 \arrow[l] & U_0 \cap U_2 \arrow[uul] \arrow[ddl] & U_0 \cap U_1 \cap U_2 \arrow[l] \arrow[uul] \arrow[ddl] \\
 & \oplus & \oplus & \\
 & U_2 \arrow[uul] & U_1 \cap U_2 \arrow[l] \arrow[uul] &
\end{tikzcd}
$$

Figure 1: Cover of $\mathbb{F}_1[X_0, X_1, X_2]$ in category $\mathcal{D}$.

$$
\begin{tikzcd}
 & V_0 \arrow[r] \arrow[ddr] & V_{0,1} \arrow[ddr] & \\
 & \oplus & \oplus & \\
\mathbb{F}_1[X_0, X_1, X_2] \arrow[uur] \arrow[r] \arrow[ddr] & V_1 \arrow[uur] \arrow[ddr] & V_{0,2} \arrow[r] & V_{0,1,2} \\
 & \oplus & \oplus & \\
 & V_2 \arrow[r] \arrow[uur] & V_{1,2} \arrow[uur] &
\end{tikzcd}
$$

Figure 2: Contravariant Category