Lie Algebra Representations and Related Geometry via the Alcove Model

Cristian Lenart

State University of New York at Albany

AMS Meeting, SUNY Buffalo, September 2023

Building Bridges Between $\mathbb{F}_1$-Geometry, Combinatorics and Representation Theory, I

Contains joint work with Alexander Postnikov (MIT), Satoshi Naito (Tokyo Inst. of Technology), Daisuke Sagaki (Tsukuba Univ.), Anne Schilling (Univ. California Davis), and Mark Shimozono (Virginia Tech).

C. Lenart was supported by the NSF grant DMS-1855592.

Tits' idea (1956): analogy between the representation theory of the symmetric group S_n and the Chevalley group $GL_n(\mathbb{F}_q)$ should find an explanation by interpreting S_n as a Chevalley group (finite group of Lie type) over the "field of characteristic one."

Tits' idea (1956): analogy between the representation theory of the symmetric group S_n and the Chevalley group $GL_n(\mathbb{F}_q)$ should find an explanation by interpreting S_n as a Chevalley group (finite group of Lie type) over the "field of characteristic one."

Chevalley groups over $\mathbb{F}_1$ and related $\mathbb{F}_1$-geometries were developed in several papers, notably:

Tits' idea (1956): analogy between the representation theory of the symmetric group S_n and the Chevalley group $GL_n(\mathbb{F}_q)$ should find an explanation by interpreting S_n as a Chevalley group (finite group of Lie type) over the "field of characteristic one."

Chevalley groups over $\mathbb{F}_1$ and related $\mathbb{F}_1$-geometries were developed in several papers, notably:

- Alain Connes and Caterina Consani. On the notion of geometry over $\mathbb{F}_1$. *J. Algebraic Geom.* 20 (2011), no. 3, 525–557.
- Oliver Lorscheid. A blueprinted view on $\mathbb{F}_1$-geometry. *Absolute arithmetic and $\mathbb{F}_1$-geometry*, 161–219. European Mathematical Society (EMS), Zürich, 2016.
- Koen Thas. The Weyl functor – introduction to absolute arithmetic. *Absolute arithmetic and $\mathbb{F}_1$-geometry*, 3–36. European Mathematical Society (EMS), Zürich, 2016.

Plan of the talk

- uniform combinatorial model for representations of complex semisimple Lie algebras (used to define the corresponding Chevalley groups), and beyond: the alcove model;

Plan of the talk

- uniform combinatorial model for representations of complex semisimple Lie algebras (used to define the corresponding Chevalley groups), and beyond: the alcove model;
- applications to Chevalley groups (Hall-Littlewood polynomials);

Plan of the talk

- uniform combinatorial model for representations of complex semisimple Lie algebras (used to define the corresponding Chevalley groups), and beyond: the alcove model;
- applications to Chevalley groups (Hall-Littlewood polynomials);
- related geometry of flag manifolds (applications to multiplication formulas in their K-theory).

Complex simple and affine Lie algebras

Main example. Type A_{n-1} ($n \geq 1$):

$$\mathfrak{sl}_n = \{\, n \text{ by } n \text{ complex matrices of trace } 0 \,\}\,, \quad [A, B] = AB - BA\,.$$

Complex simple and affine Lie algebras

Main example. Type A_{n-1} $(n \geq 1)$:

$\mathfrak{sl}_n = \{\, n$ by n complex matrices of trace $0\,\}$, $\quad [A, B] = AB - BA\,.$

The other simple Lie algebras $\mathfrak{g}$

- type B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}$;
- type C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}$;
- type D_n $(n \geq 4)$: $\mathfrak{so}_{2n}$;
- exceptional types: E_6, E_7, E_8, F_4, G_2.

Complex simple and affine Lie algebras

Main example. Type A_{n-1} $(n \geq 1)$:

$$\mathfrak{sl}_n = \{\, n \text{ by } n \text{ complex matrices of trace } 0 \,\}\,, \quad [A, B] = AB - BA\,.$$

The other simple Lie algebras $\mathfrak{g}$

- type B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}$;
- type C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}$;
- type D_n $(n \geq 4)$: $\mathfrak{so}_{2n}$;
- exceptional types: E_6, E_7, E_8, F_4, G_2.

Their (untwisted) affine versions $\widehat{\mathfrak{g}}$, of type $A_{n-1}^{(1)}, \ldots, G_2^{(1)}$.

Complex simple and affine Lie algebras

Main example. Type A_{n-1} $(n \geq 1)$:

$\mathfrak{sl}_n = \{\, n$ by n complex matrices of trace $0 \,\}$, $\quad [A, B] = AB - BA\,.$

The other simple Lie algebras $\mathfrak{g}$

- type B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}$;
- type C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}$;
- type D_n $(n \geq 4)$: $\mathfrak{so}_{2n}$;
- exceptional types: E_6, E_7, E_8, F_4, G_2.

Their (untwisted) affine versions $\widehat{\mathfrak{g}}$, of type $A_{n-1}^{(1)}, \ldots, G_2^{(1)}$.

Chevalley generators: e_i, f_i
($i = 1, \ldots, r$ for $\mathfrak{g}$ of type X_r, and $i = 0, \ldots, r$ for $\widehat{\mathfrak{g}}$ of type $X_r^{(1)}$).

Representations of Lie algebras

Definition. A complex vector space V is a representation of $\mathfrak{g} \iff$ each $X \in \mathfrak{g}$ is represented as a linear map $X : V \to V$, compatibly with the Lie bracket.

Representations of Lie algebras

Definition. A complex vector space V is a representation of $\mathfrak{g} \iff$ each $X \in \mathfrak{g}$ is represented as a linear map $X : V \to V$, compatibly with the Lie bracket.

Irreducible representation: $V \neq V_1 \oplus V_2$.

Representations of Lie algebras

Definition. A complex vector space V is a representation of $\mathfrak{g} \iff$ each $X \in \mathfrak{g}$ is represented as a linear map $X : V \to V$, compatibly with the Lie bracket.

Irreducible representation: $V \neq V_1 \oplus V_2$.

Fact. The irreducible representations of the (semi)simple and affine Lie algebras are indexed by certain vectors λ (highest/dominant weights), and they are denoted $V(\lambda)$.

Representations of Lie algebras

Definition. A complex vector space V is a representation of $\mathfrak{g} \iff$ each $X \in \mathfrak{g}$ is represented as a linear map $X : V \to V$, compatibly with the Lie bracket.

Irreducible representation: $V \neq V_1 \oplus V_2$.

Fact. The irreducible representations of the (semi)simple and affine Lie algebras are indexed by certain vectors λ (highest/dominant weights), and they are denoted $V(\lambda)$.

Example. In type A_{n-1}, the highest weights are partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0)$.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$, for the purpose of various computations ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$, for the purpose of various computations ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

Kashiwara (crystal) operators are modified versions of the Chevalley generators: $\widetilde{e}_i$, $\widetilde{f}_i$.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$, for the purpose of various computations ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

Kashiwara (crystal) operators are modified versions of the Chevalley generators: $\widetilde{e}_i$, $\widetilde{f}_i$.

Fact. V has a crystal basis $B \implies$ in the limit $q \to 0$ we have

$$\widetilde{f}_i, \widetilde{e}_i \;:\; B \to B \sqcup \{0\}\,,$$

$$\widetilde{f}_i\, b = b' \iff \widetilde{e}_i\, b' = b \iff b \xrightarrow{i} b'\,.$$

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$, for the purpose of various computations ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

Kashiwara (crystal) operators are modified versions of the Chevalley generators: $\widetilde{e}_i$, $\widetilde{f}_i$.

Fact. V has a crystal basis $B \implies$ in the limit $q \to 0$ we have

$$\widetilde{f}_i, \widetilde{e}_i \;:\; B \to B \sqcup \{0\}\,,$$

$$\widetilde{f}_i\, b = b' \quad \Longleftrightarrow \quad \widetilde{e}_i\, b' = b \quad \Longleftrightarrow \quad b \stackrel{i}{\to} b'\,.$$

Crystal graph: directed graph on B with arrows colored i.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$, for the purpose of various computations ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

Kashiwara (crystal) operators are modified versions of the Chevalley generators: $\widetilde{e}_i$, $\widetilde{f}_i$.

Fact. V has a crystal basis $B \implies$ in the limit $q \to 0$ we have

$$\widetilde{f}_i, \widetilde{e}_i \;:\; B \to B \sqcup \{0\}\,,$$

$$\widetilde{f}_i\, b = b' \iff \widetilde{e}_i\, b' = b \iff b \xrightarrow{i} b'\,.$$

Crystal graph: directed graph on B with arrows colored i.

Fact. The irreducible representations $V(\lambda)$ have associated crystals $B(\lambda)$.

Type A_{n-1}

Fact. The vertices of $B(\lambda)$ can be labeled by semistandard Young tableaux (SSYT) of shape λ, filled with $\{1, \ldots, n\}$:

Type A_{n-1}

Fact. The vertices of $B(\lambda)$ can be labeled by semistandard Young tableaux (SSYT) of shape λ, filled with $\{1, \ldots, n\}$:

$$n = 5\,,\ \lambda = (4,2,2,1)\,, \qquad b = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\,.$$

Type A_{n-1}

Fact. The vertices of $B(\lambda)$ can be labeled by semistandard Young tableaux (SSYT) of shape λ, filled with $\{1, \ldots, n\}$:

$$n = 5\,,\ \lambda = (4,2,2,1)\,, \qquad b = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\,.$$

Crystal operators: $\widetilde{f}_i$ changes a given i into $i+1$.

Type A_{n-1}

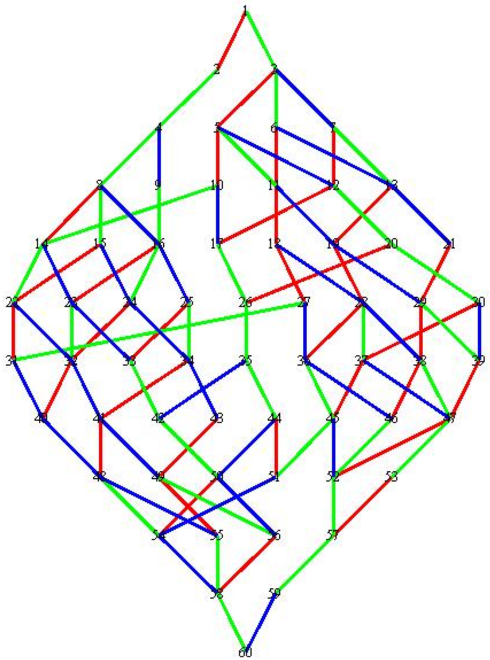
Fact. The vertices of $B(\lambda)$ can be labeled by semistandard Young tableaux (SSYT) of shape λ, filled with $\{1, \ldots, n\}$:

$$n = 5\,,\;\; \lambda = (4, 2, 2, 1)\,, \qquad b = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \;.$$

Crystal operators: $\widetilde{f}_i$ changes a given i into $i + 1$.

Types $B - D$: Kashiwara-Nakashima tableaux (contain negative entries as well).

$n = 4$, $\lambda = (3, 3, 1)$, blue: $1 \to 2$, green: $2 \to 3$, red: $3 \to 4$.



Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations of affine Lie algebras $\widehat{\mathfrak{g}}$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations of affine Lie algebras $\widehat{\mathfrak{g}}$.

They are the building blocks for the (infinite) highest weight crystals $B(\Lambda)$, constructed as infinite tensor products of KR crystals.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations of affine Lie algebras $\widehat{\mathfrak{g}}$.

They are the building blocks for the (infinite) highest weight crystals $B(\Lambda)$, constructed as infinite tensor products of KR crystals.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations of affine Lie algebras $\widehat{\mathfrak{g}}$.

They are the building blocks for the (infinite) highest weight crystals $B(\Lambda)$, constructed as infinite tensor products of KR crystals.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, so they are denoted $B^{p,q}$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations of affine Lie algebras $\widehat{\mathfrak{g}}$.

They are the building blocks for the (infinite) highest weight crystals $B(\Lambda)$, constructed as infinite tensor products of KR crystals.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, so they are denoted $B^{p,q}$. We only consider column shapes $B^{p,1}$.

Tensor products of KR crystals

Let $\mathbf{p} = (p_1, \ldots, p_m)$ and

$$B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}\,.$$

Tensor products of KR crystals

Let $\mathbf{p} = (p_1, \ldots, p_m)$ and

$$B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes \ldots \otimes B^{p_m,1} .$$

Remark. The tensor product of crystals is constructed via a specific rule, which corresponds to the tensor product of representations.

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Consider

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}\,.$$

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Consider

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}\,.$$

$B^{k,1} \simeq B(1^k)$,

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Consider

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}\,.$$

$B^{k,1} \simeq B(1^k)$, so the vertices of $B^{\otimes \mathbf{p}}$ are represented as column-strict fillings of the diagram $\mathbf{p}$ with integers $1, \ldots, n$.

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Consider

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \ldots \otimes B^{p_m,1} \,.$$

$B^{k,1} \simeq B(1^k)$, so the vertices of $B^{\otimes \mathbf{p}}$ are represented as column-strict fillings of the diagram $\mathbf{p}$ with integers $1, \ldots, n$.

Example. $n = 5$

$$b = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 1 & 4 \\ \hline 3 & \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{1-1} \cline{3-4} 5 & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \cline{3-3} \end{array} \in B^{3,1} \otimes B^{1,1} \otimes B^{3,1} \otimes B^{2,1} \,.$$

Example in type $A_{n-1}^{(1)}$

View $\mathbf{p} = (p_1, \ldots, p_m)$ as a diagram with *columns* of heights $p_1, \ldots, p_m$.

Consider

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}\,.$$

$B^{k,1} \simeq B(1^k)$, so the vertices of $B^{\otimes \mathbf{p}}$ are represented as column-strict fillings of the diagram $\mathbf{p}$ with integers $1, \ldots, n$.

Example. $n = 5$

$$b = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 1 & 4 \\ \hline 3 & & 3 & 5 \\ \hline 5 & & 4 & \\ \hline \end{array} \in B^{3,1} \otimes B^{1,1} \otimes B^{3,1} \otimes B^{2,1}\,.$$

Fact. The crystal operators $\widetilde{f}_i$, $i = 0, \ldots, n-1$, are defined by the tensor product rule, based on their action on a column:

$$1 \xrightarrow{\widetilde{f}_1} 2 \xrightarrow{\widetilde{f}_2} 3 \ldots n-1 \xrightarrow{\widetilde{f}_{n-1}} n \xrightarrow{\widetilde{f}_0} 1\,.$$

Main goal

Goal. Generalize the tableau models in classical types $A - D$ by constructing a **uniform** combinatorial model for crystals of finite and affine type.

Main goal

Goal. Generalize the tableau models in classical types $A - D$ by constructing a **uniform** combinatorial model for crystals of finite and affine type.

The uniform model: alcove model.

Main goal

Goal. Generalize the tableau models in classical types $A - D$ by constructing a **uniform** combinatorial model for crystals of finite and affine type.

The uniform model: alcove model.

Setup. The corresponding finite root system (of type $A_{n-1} - G_2$) and the associated alcove picture.

Main goal

Goal. Generalize the tableau models in classical types $A - D$ by constructing a **uniform** combinatorial model for crystals of finite and affine type.

The uniform model: alcove model.

Setup. The corresponding finite root system (of type $A_{n-1} - G_2$) and the associated alcove picture.

Main idea. Replace tableau combinatorics (insertion, jeu de taquin, charge, etc.) with combinatorics of finite Weyl groups / reflection groups (reduced decompositions, Bruhat order, etc.)

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle\alpha, \alpha\rangle$.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Fundamental weights: ω_i, where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Fundamental weights: ω_i, where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Weights / weight lattice: $P := \{\sum_i c_i \omega_i \ : \ c_i \in \mathbb{Z}\}$;
dominant weights P^+ if $c_i \geq 0$.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Fundamental weights: ω_i, where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Weights / weight lattice: $P := \{\sum_i c_i \omega_i \ : \ c_i \in \mathbb{Z}\}$;
dominant weights P^+ if $c_i \geq 0$.

Example. Type A_{n-1}.

$V = \mathbb{R}^n / \langle \varepsilon_1 + \ldots + \varepsilon_n \rangle$ $(r = n - 1)$.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Fundamental weights: ω_i, where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Weights / weight lattice: $P := \{\sum_i c_i \omega_i \ : \ c_i \in \mathbb{Z}\}$;
dominant weights P^+ if $c_i \geq 0$.

Example. Type A_{n-1}.

$V = \mathbb{R}^n / \langle \varepsilon_1 + \ldots + \varepsilon_n \rangle$ $(r = n - 1)$.

$\Phi = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j) \ : \ 1 \leq i \neq j \leq n\}$.

Background on finite root systems

$\Phi \subset V = \mathbb{R}^r$ invariant under reflections s_α, $\alpha \in \Phi$, in the hyperplane orthogonal to α.

Coroot: $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Simple roots: $\alpha_1, \ldots, \alpha_r \in \Phi$; form a basis of V.

Fundamental weights: ω_i, where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Weights / weight lattice: $P := \{\sum_i c_i\omega_i \ : \ c_i \in \mathbb{Z}\}$;
dominant weights P^+ if $c_i \geq 0$.

Example. Type A_{n-1}.

$V = \mathbb{R}^n/\langle \varepsilon_1 + \ldots + \varepsilon_n \rangle$ $(r = n-1)$.

$\Phi = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i,j) \ : \ 1 \leq i \neq j \leq n\}$.

Weights / dominant weights / fundamental weights:
compositions / partitions / columns $\omega_k = (1^k) = \varepsilon_1 + \ldots + \varepsilon_k$.

The Weyl group

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle \, .$$

The Weyl group

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle \, .$$

Length: $\ell(w) = \min \{ k \ : \ w = s_{i_1} \ldots s_{i_k} \} \, .$

The Weyl group

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle \, .$$

Length: $\ell(w) = \min \{ k \ : \ w = s_{i_1} \ldots s_{i_k} \} \, .$

Example. Type A_{n-1}.

$W = S_n \, , \ \ s_{\varepsilon_i - \varepsilon_j}$ is the transposition $t_{ij} \, , \, s_i = t_{i,i+1} \, .$

The Weyl group

$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle \, .$

Length: $\ell(w) = \min \{ k \ : \ w = s_{i_1} \ldots s_{i_k} \} \, .$

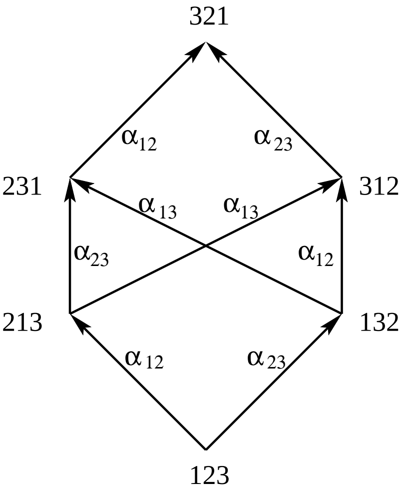
Example. Type A_{n-1}.

$W = S_n \, , \quad s_{\varepsilon_i - \varepsilon_j}$ is the transposition t_{ij} , $s_i = t_{i,i+1}$.

The Bruhat order on W has covers

$$w \lessdot w s_\alpha \, , \qquad \text{where } \ell(w s_\alpha) = \ell(w) + 1 \, .$$

Bruhat graph for S_3:



Alcove picture

Hyperplanes $H_{\alpha,k} = \{\lambda : \langle \lambda, \alpha^\vee \rangle = k\}$ ($k \in \mathbb{Z}$), affine reflections $s_{\alpha,k}$.

Alcove picture

Hyperplanes $H_{\alpha,k} = \{\lambda \,:\, \langle \lambda, \alpha^\vee \rangle = k\}$ ($k \in \mathbb{Z}$), affine reflections $s_{\alpha,k}$.

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k}\right)$.

Alcove picture

Hyperplanes $H_{\alpha,k} = \{\lambda \,:\, \langle \lambda, \alpha^\vee \rangle = k\}$ ($k \in \mathbb{Z}$), affine reflections $s_{\alpha,k}$.

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k}\right)$.

Fundamental alcove $A_\circ$ with a vertex at 0.

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_m)$.

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_m)$.

Definition. Indexing set $\mathcal{A}(\Gamma)$ for a basis of $V(\lambda)$, i.e., the vertices of $B(\lambda)$, consists of:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$$

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_m)$.

Definition. Indexing set $\mathcal{A}(\Gamma)$ for a basis of $V(\lambda)$, i.e., the vertices of $B(\lambda)$, consists of:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$$

such that we have the following saturated chain in Bruhat order:

$$Id \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s}\,.$$

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_m)$.

Definition. Indexing set $\mathcal{A}(\Gamma)$ for a basis of $V(\lambda)$, i.e., the vertices of $B(\lambda)$, consists of:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$$

such that we have the following saturated chain in Bruhat order:

$$Id \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s}\,.$$

Such subsets will be called admissible subsets. Geometrically, they give folded alcove paths or alcove walks, denoted $\Gamma(J)$.

Alcove paths/walks

Definition. Given $\lambda \in P^+$, an alcove path is a shortest sequence of adjacent alcoves

$$(A_\circ = A_0, A_1, \ldots, A_m = A_\circ - \lambda)\,.$$

A_{i-1} and A_i separated by a hyperplane H_{β_i, l_i} orthogonal to β_i.

Let $r_i := s_{\beta_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_m)$.

Definition. Indexing set $\mathcal{A}(\Gamma)$ for a basis of $V(\lambda)$, i.e., the vertices of $B(\lambda)$, consists of:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$$

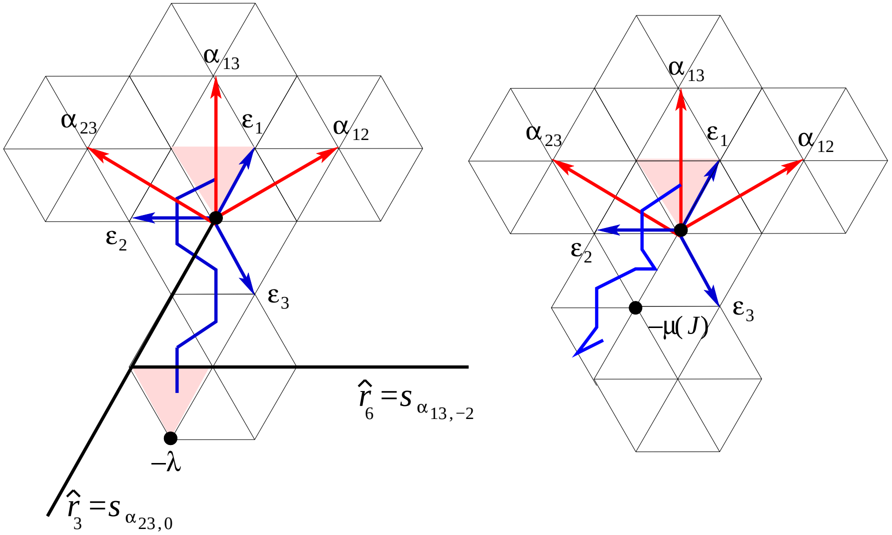
such that we have the following saturated chain in Bruhat order:

$$Id \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s}\,.$$

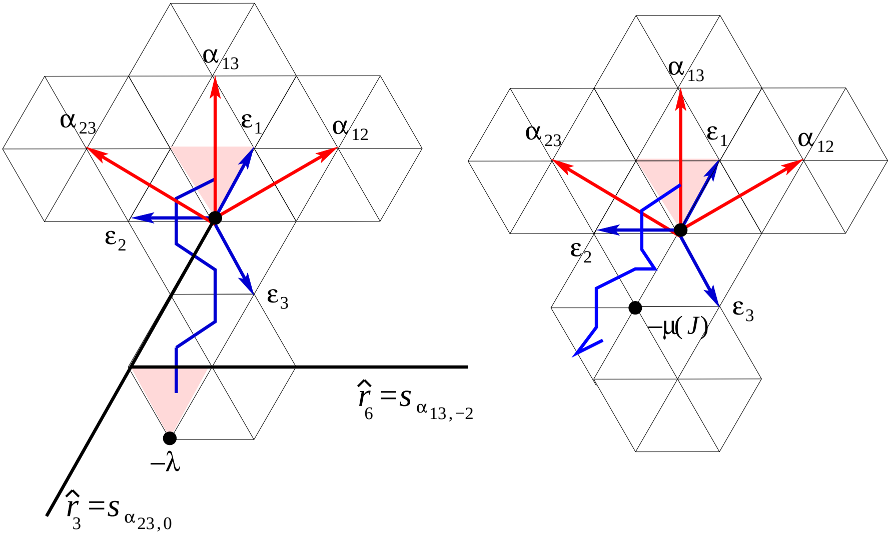
Such subsets will be called admissible subsets. Geometrically, they give folded alcove paths or alcove walks, denoted $\Gamma(J)$.

Weight of an admissible subset: $\mu(J) = -$endpoint of $\Gamma(J)$.

Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\ )$.



Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\,(1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\,)$.



$J = \{3,6\}$, saturated chain: $Id = 123 \lessdot t_{23} = 132 \lessdot t_{23}t_{13} = 231$.

Applications to irreducible characters of semisimple Lie algebras

Fix a λ-chain Γ.

Theorem. *The irreducible character $ch(V(\lambda))$ of $\mathfrak{g}$ can be expressed as*

$$ch(V(\lambda)) = \sum_{J \in \mathcal{A}(\Gamma)} x^{\mu(J)}\,.$$

Applications to irreducible characters of semisimple Lie algebras

Fix a λ-chain Γ.

Theorem. *The irreducible character $ch(V(\lambda))$ of $\mathfrak{g}$ can be expressed as*

$$ch(V(\lambda)) = \sum_{J \in \mathcal{A}(\Gamma)} x^{\mu(J)}\,.$$

Theorem. [L.-Postnikov] *Let $\Gamma(J)$ be the alcove walk corresponding to J. We have the Littlewood-Richardson rule:*

$$ch(V(\lambda)) \cdot ch(V(\nu)) = \sum_{J} ch(V(\nu + \mu(J)))\,,$$

where the summation is over all $J \in \mathcal{A}(\Gamma)$ s.t.

$$\Gamma(J) + \nu + \mu(J) \subseteq P_{\mathbb{R}}^{+}\,.$$

The crystal structure

Theorem. [L.-Postnikov] *The structure of the crystal* $B(\lambda)$ *is realized on the set* $\mathcal{A}(\Gamma)$ *by combinatorial crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$*:*

$$\widetilde{f}_i(J) := \begin{cases} J \cup \{l\} \\ J \setminus \{k\} \cup \{l\} \,. \end{cases}$$

The crystal structure

Theorem. [L.-Postnikov] *The structure of the crystal $B(\lambda)$ is realized on the set $\mathcal{A}(\Gamma)$ by combinatorial crystal operators $\widetilde{f}_1, \ldots, \widetilde{f}_r$:*

$$\widetilde{f}_i(J) := \begin{cases} J \cup \{l\} \\ J \setminus \{k\} \cup \{l\}\,. \end{cases}$$

Remarks. (1) The model is independent of the λ-chain Γ, so we will use the notation $\mathcal{A}(\lambda) = \mathcal{A}(\Gamma)$. This will be the case with other variations of the alcove model.

The crystal structure

Theorem. [L.-Postnikov] *The structure of the crystal* $B(\lambda)$ *is realized on the set* $\mathcal{A}(\Gamma)$ *by combinatorial crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$*:*

$$\widetilde{f}_i(J) := \begin{cases} J \cup \{l\} \\ J \setminus \{k\} \cup \{l\}\,. \end{cases}$$

Remarks. (1) The model is independent of the λ-chain Γ, so we will use the notation $\mathcal{A}(\lambda) = \mathcal{A}(\Gamma)$. This will be the case with other variations of the alcove model.

(2) We generalized the mentioned results to symmetrizable Kac-Moody algebras.

The realization of Kirillov-Reshetikhin (affine) crystals: the quantum Bruhat graph

Consider $(W, \leq)$, with covers $w \lessdot ws_\alpha$ thought of as labeled directed edges $w \xrightarrow{\alpha} ws_\alpha$.

The realization of Kirillov-Reshetikhin (affine) crystals: the quantum Bruhat graph

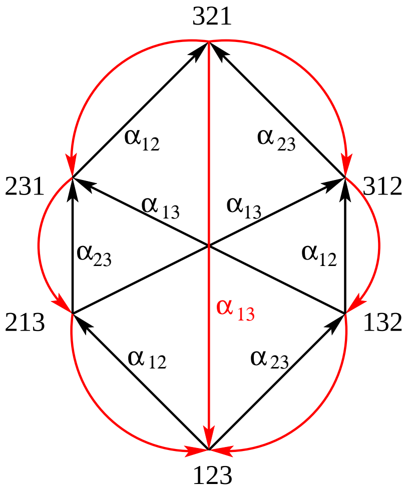
Consider $(W, \leq)$, with covers $w \lessdot ws_\alpha$ thought of as labeled directed edges $w \xrightarrow{\alpha} ws_\alpha$.

Definition. The quantum Bruhat graph on W, denoted $\mathrm{QBG}(W)$, is defined by adding downward edges

$$w \xrightarrow{\alpha} ws_\alpha\,, \qquad \text{where } \ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,.$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

Quantum Bruhat graph for S_3:



The realization of KR crystals (cont.)

Definition. Given $\lambda \in P^+$ and a λ-chain $\Gamma = (\beta_1, \ldots, \beta_m)$, consider the set $\mathcal{A}_q(\lambda) = \mathcal{A}_q(\Gamma)$ of

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}\,,$$

The realization of KR crystals (cont.)

Definition. Given $\lambda \in P^+$ and a λ-chain $\Gamma = (\beta_1, \ldots, \beta_m)$, consider the set $\mathcal{A}_q(\lambda) = \mathcal{A}_q(\Gamma)$ of

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}\,,$$

given by the following weaker admissibility condition:

$$Id \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s}\,;$$

recall the notation $r_i := s_{\beta_i}$.

The realization of KR crystals (cont.)

Definition. Given $\lambda \in P^+$ and a λ-chain $\Gamma = (\beta_1, \ldots, \beta_m)$, consider the set $\mathcal{A}_q(\lambda) = \mathcal{A}_q(\Gamma)$ of

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\},$$

given by the following weaker admissibility condition:

$$Id \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s};$$

recall the notation $r_i := s_{\beta_i}$.

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] *Given* $(p_1, \ldots, p_m)$ *and an arbitrary Lie type, let*

$$\lambda = \omega_{p_1} + \ldots + \omega_{p_m}.$$

The realization of KR crystals (cont.)

Definition. Given $\lambda \in P^+$ and a λ-chain $\Gamma = (\beta_1, \ldots, \beta_m)$, consider the set $\mathcal{A}_q(\lambda) = \mathcal{A}_q(\Gamma)$ of

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}\,,$$

given by the following weaker admissibility condition:

$$Id \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s}\,;$$

recall the notation $r_i := s_{\beta_i}$.

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] *Given* $(p_1, \ldots, p_m)$ *and an arbitrary Lie type, let*

$$\lambda = \omega_{p_1} + \ldots + \omega_{p_m}\,.$$

The crystal structure of the tensor product of KR crystals $B^{p_1,1} \otimes \ldots \otimes B^{p_m,1}$ *is realized on the set* $\mathcal{A}_q(\lambda)$ *by combinatorial crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$ *and* $\widetilde{f}_0$.

Macdonald polynomials

λ: dominant weight for a finite root system.

Macdonald polynomials

λ: dominant weight for a finite root system.

$P_\lambda(X; q, t)$: Weyl group invariant polynomials, orthogonal, generalizing the corresponding irreducible characters

$$ch(V(\lambda)) = P_\lambda(X; 0, 0)\,.$$

Macdonald polynomials

λ: dominant weight for a finite root system.

$P_\lambda(X; q, t)$: Weyl group invariant polynomials, orthogonal, generalizing the corresponding irreducible characters

$$ch(V(\lambda)) = P_\lambda(X; 0, 0)\,.$$

Deep connections with:

- affine Lie algebras
- double affine Hecke algebras
- Hilbert schemes
- quantum integrable systems
- conformal field theory
- etc.

Applications of the alcove model for KR crystals

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] The following were proved uniformly in arbitrary Lie type.

Applications of the alcove model for KR crystals

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] The following were proved uniformly in arbitrary Lie type.

- The graded character of a tensor product of KR crystals (grading by the energy function) coincides with a specialized Macdonald polynomial $P_\lambda(X; q,\ t = 0)$.

Applications of the alcove model for KR crystals

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] The following were proved uniformly in arbitrary Lie type.

- The graded character of a tensor product of KR crystals (grading by the energy function) coincides with a specialized Macdonald polynomial $P_\lambda(X; q,\ t = 0)$.
- Combinatorial formula for the energy function.

Applications of the alcove model for KR crystals

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] The following were proved uniformly in arbitrary Lie type.

- The graded character of a tensor product of KR crystals (grading by the energy function) coincides with a specialized Macdonald polynomial $P_\lambda(X; q,\ t = 0)$.
- Combinatorial formula for the energy function.
- Algorithm for computing the combinatorial R-matrix (unique crystal isomorphism commuting factors in a tensor product of KR crystals).

Hall-Littlewood polynomials $P_\lambda(X;t)$

$P_\lambda(X;t)$ is a specialized Macdonald polynomial $P_\lambda(X;\, q=0,\, t)$.

Hall-Littlewood polynomials $P_\lambda(X;t)$

$P_\lambda(X;t)$ is a specialized Macdonald polynomial $P_\lambda(X;\ q=0,\ t)$.

$P_\lambda(X;t)$ is usually defined by a t-analogue of the Weyl character formula (for the irreducible characters).

Hall-Littlewood polynomials $P_\lambda(X; t)$ (cont.)

The Hall-Littlewood polynomials are related to:

- Enumerative properties of finite abelian groups, Hall algebra.

Hall-Littlewood polynomials $P_\lambda(X;t)$ (cont.)

The Hall-Littlewood polynomials are related to:

- Enumerative properties of finite abelian groups, Hall algebra.
- Enumeration of rational points over a finite field of the unipotent partial flag variety.

Hall-Littlewood polynomials $P_\lambda(X;t)$ (cont.)

The Hall-Littlewood polynomials are related to:

- Enumerative properties of finite abelian groups, Hall algebra.
- Enumeration of rational points over a finite field of the unipotent partial flag variety.
- The character theory of general linear groups over finite fields.

Hall-Littlewood polynomials $P_\lambda(X;t)$ (cont.)

The Hall-Littlewood polynomials are related to:

- Enumerative properties of finite abelian groups, Hall algebra.
- Enumeration of rational points over a finite field of the unipotent partial flag variety.
- The character theory of general linear groups over finite fields.
- The theory of p-adic groups (spherical functions) and Hecke algebras.

Hall-Littlewood polynomials $P_\lambda(X; t)$ (cont.)

The Hall-Littlewood polynomials are related to:

- Enumerative properties of finite abelian groups, Hall algebra.
- Enumeration of rational points over a finite field of the unipotent partial flag variety.
- The character theory of general linear groups over finite fields.
- The theory of p-adic groups (spherical functions) and Hecke algebras.
- Lusztig's t-analogues of weight multiplicities (Kostka-Foulkes polynomials).
- etc.

Applications of the alcove model to $P_\lambda(X;t)$

Given $\lambda \in P^+$, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Applications of the alcove model to $P_\lambda(X;t)$

Given $\lambda \in P^+$, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Definition. Let $\mathcal{A}_<(\lambda)$ be the set of admissible pairs (w, J), i.e., pairs satisfying

$$w \in W\,, \quad J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\} \quad \text{and}$$
$$w < wr_{j_1} < \ldots < wr_{j_1} \ldots r_{j_s} \in W^\lambda \quad \text{not necessarily covers}\,.$$

Applications of the alcove model to $P_\lambda(X;t)$

Given $\lambda \in P^+$, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Definition. Let $\mathcal{A}_<(\lambda)$ be the set of admissible pairs (w, J), i.e., pairs satisfying

$$w \in W\,, \quad J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\} \quad \text{and}$$
$$w < wr_{j_1} < \ldots < wr_{j_1} \ldots r_{j_s} \in W^\lambda \quad \text{not necessarily covers}\,.$$

Theorem. [Schwer, Ram] (1) *We have*

$$P_\lambda(X;t) = \sum_{(w,J)\in\mathcal{A}_<(\lambda)} t^{\frac{1}{2}(\ell(w)+\ell(w\phi(J))-|J|)}\,(1-t)^{|J|}\,x^{w(\mu(J))}\,.$$

Applications of the alcove model to $P_\lambda(X;t)$

Given $\lambda \in P^+$, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Definition. Let $\mathcal{A}_<(\lambda)$ be the set of admissible pairs (w, J), i.e., pairs satisfying

$$w \in W\,, \quad J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\} \quad \text{and}$$
$$w < wr_{j_1} < \ldots < wr_{j_1} \ldots r_{j_s} \in W^\lambda \quad \text{not necessarily covers}\,.$$

Theorem. [Schwer, Ram] (1) *We have*

$$P_\lambda(X;t) = \sum_{(w,J)\in\mathcal{A}_<(\lambda)} t^{\frac{1}{2}(\ell(w)+\ell(w\phi(J))-|J|)}\,(1-t)^{|J|}\,x^{w(\mu(J))}\,.$$

(2) *There is also a similar Littlewood-Richardson rule for expanding the product of two Hall-Littlewood polynomials.*

Question related to $\mathbb{F}_1$

Facts. (1) [Gaussent-Littelmann] The alcove model is closely related to galleries in affine buildings (via the affine Grassmannian).

Question related to $\mathbb{F}_1$

Facts. (1) [Gaussent-Littelmann] The alcove model is closely related to galleries in affine buildings (via the affine Grassmannian).

(2) [Lorscheid, Thas] There is a theory of buildings over $\mathbb{F}_1$ and an action of the corresponding Chevalley groups over $\mathbb{F}_1$ on them.

Question related to $\mathbb{F}_1$

Facts. (1) [Gaussent-Littelmann] The alcove model is closely related to galleries in affine buildings (via the affine Grassmannian).

(2) [Lorscheid, Thas] There is a theory of buildings over $\mathbb{F}_1$ and an action of the corresponding Chevalley groups over $\mathbb{F}_1$ on them.

Question. Alcove model over $\mathbb{F}_1$ and its applications to representations of Chevalley groups over $\mathbb{F}_1$?

The connection to geometry

$$
\begin{tikzcd}
\text{representation theory of semisimple Lie algebras / Lie groups } G \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{geometry of flag manifolds } G/B \arrow[dl, leftrightarrow] \\
& \text{combinatorics} &
\end{tikzcd}
$$

Geometric setup

Complex semisimple Lie group G.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Structure sheaves of Schubert varieties: $\mathcal{O}_w = \mathcal{O}_{X_w}$.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Structure sheaves of Schubert varieties: $\mathcal{O}_w = \mathcal{O}_{X_w}$.

Line bundle: for $\lambda \in P$, let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, with $\mathbb{C}_{-\lambda} = \mathbb{C}$.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Structure sheaves of Schubert varieties: $\mathcal{O}_w = \mathcal{O}_{X_w}$.

Line bundle: for $\lambda \in P$, let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, with $\mathbb{C}_{-\lambda} = \mathbb{C}$.

Example. Type A_{n-1}.

$G = SL_n(\mathbb{C})$, $B = \{\,\text{upper triangular matrices in } SL_n\,\}$, $W = S_n$.

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Structure sheaves of Schubert varieties: $\mathcal{O}_w = \mathcal{O}_{X_w}$.

Line bundle: for $\lambda \in P$, let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, with $\mathbb{C}_{-\lambda} = \mathbb{C}$.

Example. Type A_{n-1}.

$G = SL_n(\mathbb{C})$, $B = \{$ upper triangular matrices in $SL_n \}$, $W = S_n$.

Classical flag manifold

$$SL_n/B = Fl_n = \{(0 \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n)\}.$$

Geometric setup

Complex semisimple Lie group G.

Borel subgroup $B \subset G$, maximal torus $T \subset B$, opposite Borel B_-, parabolic subgroup P.

Generalized flag manifold G/B, partial flag manifold G/P.

Schubert varieties $X_w = \overline{BwB_-/B_-}$, for $w \in W$.

Structure sheaves of Schubert varieties: $\mathcal{O}_w = \mathcal{O}_{X_w}$.

Line bundle: for $\lambda \in P$, let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, with $\mathbb{C}_{-\lambda} = \mathbb{C}$.

Example. Type A_{n-1}.

$G = SL_n(\mathbb{C})$, $B = \{\text{upper triangular matrices in } SL_n\}$, $W = S_n$.

Classical flag manifold

$$SL_n/B = Fl_n = \{(0 \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n)\}.$$

Partial flag manifold: Grassmannian of k-subspaces in $\mathbb{C}^n$.

Geometric setup (cont.)

$$\mathbb{Z}[X] := \langle x^\lambda \ : \ \lambda \in P \rangle \, ; \quad \mathbb{Z}[P] := \langle e^\lambda \ : \ \lambda \in P \rangle \, .$$

Geometric setup (cont.)

$$\mathbb{Z}[X] := \langle x^\lambda \;:\; \lambda \in P \rangle\,; \quad \mathbb{Z}[P] := \langle e^\lambda \;:\; \lambda \in P \rangle\,.$$

Grothendieck ring of coherent T-equivariant sheaves $K_T(G/B)$.

Geometric setup (cont.)

$$\mathbb{Z}[X] := \langle x^\lambda \ : \ \lambda \in P \rangle \, ; \quad \mathbb{Z}[P] := \langle e^\lambda \ : \ \lambda \in P \rangle \, .$$

Grothendieck ring of coherent T-equivariant sheaves $K_T(G/B)$.

Fact.

$$K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[P])/\mathcal{I} \quad \text{via} \quad [\mathcal{L}_\lambda] \mapsto e^\lambda \, .$$

Geometric setup (cont.)

$$\mathbb{Z}[X] := \langle x^\lambda \,:\, \lambda \in P \rangle\,; \quad \mathbb{Z}[P] := \langle e^\lambda \,:\, \lambda \in P \rangle\,.$$

Grothendieck ring of coherent T-equivariant sheaves $K_T(G/B)$.

Fact.

$$K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[P])/\mathcal{I} \quad \text{via} \quad [\mathcal{L}_\lambda] \mapsto e^\lambda\,.$$

Fact. [Kostant-Kumar] The Schubert classes $\{[\mathcal{O}_w] \,:\, w \in W\}$ form a basis of $K_T(G/B)$ over $\mathbb{Z}[X] = K_T(\mathrm{pt})$.

Geometric setup (cont.)

$$\mathbb{Z}[X] := \langle x^\lambda \ : \ \lambda \in P\rangle\,; \qquad \mathbb{Z}[P] := \langle e^\lambda \ : \ \lambda \in P\rangle\,.$$

Grothendieck ring of coherent T-equivariant sheaves $K_T(G/B)$.

Fact.

$$K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[P])/\mathcal{I} \quad \text{via} \quad [\mathcal{L}_\lambda] \mapsto e^\lambda\,.$$

Fact. [Kostant-Kumar] The Schubert classes $\{[\mathcal{O}_w] \ : \ w \in W\}$ form a basis of $K_T(G/B)$ over $\mathbb{Z}[X] = K_T(\mathrm{pt})$.

Chevalley-type formula for restricting $\mathcal{L}_\lambda$ to a Schubert variety X_w:

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_w] = \sum_{v \in W} c_w^v(\lambda)\,[\mathcal{O}_v]\,, \qquad c_w^v(\lambda) \in \mathbb{Z}[X]\,.$$

A combinatorial Chevalley formula

Given any $\lambda \in P$, we consider an alcove path to $A_\circ - \lambda$, and the corresponding λ-chain (of roots) $\Gamma = (\beta_1, \ldots, \beta_m)$. Recall the notation $r_i := s_{\beta_i}$.

A combinatorial Chevalley formula

Given any $\lambda \in P$, we consider an alcove path to $A_\circ - \lambda$, and the corresponding λ-chain (of roots) $\Gamma = (\beta_1, \ldots, \beta_m)$. Recall the notation $r_i := s_{\beta_i}$.

Definition. Given $w \in W$, consider the set $\mathcal{A}(w, \lambda) = \mathcal{A}(w, \Gamma)$ of $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$ given by the following admissibility condition:

$$w \lessdot wr_{j_1} \lessdot wr_{j_1} r_{j_2} \lessdot \ldots \lessdot wr_{j_1} \ldots r_{j_s} =: \pi(w, J)\,.$$

A combinatorial Chevalley formula

Given any $\lambda \in P$, we consider an alcove path to $A_\circ - \lambda$, and the corresponding λ-chain (of roots) $\Gamma = (\beta_1, \ldots, \beta_m)$. Recall the notation $r_i := s_{\beta_i}$.

Definition. Given $w \in W$, consider the set $\mathcal{A}(w, \lambda) = \mathcal{A}(w, \Gamma)$ of $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$ given by the following admissibility condition:

$$w \lessdot wr_{j_1} \lessdot wr_{j_1} r_{j_2} \lessdot \ldots \lessdot wr_{j_1} \ldots r_{j_s} =: \pi(w, J)\,.$$

Let $n(J)$ be the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

A combinatorial Chevalley formula

Given any $\lambda \in P$, we consider an alcove path to $A_\circ - \lambda$, and the corresponding λ-chain (of roots) $\Gamma = (\beta_1, \ldots, \beta_m)$. Recall the notation $r_i := s_{\beta_i}$.

Definition. Given $w \in W$, consider the set $\mathcal{A}(w, \lambda) = \mathcal{A}(w, \Gamma)$ of $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}$ given by the following admissibility condition:

$$w \lessdot wr_{j_1} \lessdot wr_{j_1} r_{j_2} \lessdot \ldots \lessdot wr_{j_1} \ldots r_{j_s} =: \pi(w, J)\,.$$

Let $n(J)$ be the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

Theorem. [L.-Postnikov] *Given the above setup, we have in $K_T(G/B)$ and $K_T(G/P)$*

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_w] = \sum_{J \in \mathcal{A}(w,\lambda)} (-1)^{n(J)} x^{w(\mu(J))} \, [\mathcal{O}_{\pi(w,J)}]\,.$$

Further work

Theorem. [L.-Naito-Sagaki] Chevalley rules for the

- T-equivariant K-theory of the semi-infinite flag manifold (non-standard affine version of the classical flag manifold G/B);

Further work

Theorem. [L.-Naito-Sagaki] Chevalley rules for the

- T-equivariant K-theory of the semi-infinite flag manifold (non-standard affine version of the classical flag manifold G/B);
- T-equivariant quantum K-theory of the classical flag manifolds G/B and G/P.

Further work

Theorem. [L.-Naito-Sagaki] Chevalley rules for the

- T-equivariant K-theory of the semi-infinite flag manifold (non-standard affine version of the classical flag manifold G/B);
- T-equivariant quantum K-theory of the classical flag manifolds G/B and G/P.

Remark. Both rules involve the alcove model based on the quantum Bruhat graph.