Moduli of Representations of Clannish Algebras

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Preliminary Assumptions

- Throughout, $k = \overline{k}$ and char $k = 0$.
- All quivers will be finite and connected.
- All algebras will be assumed to be associative and finite dimensional over k.

Context

- Moduli of representations of finite dimensional algebras were introduced by Alastair King in [King '94].
- Moduli of representations can be arbitrarily complicated [Hille '96, Huisgen-Zimmerman '98]

Conjecture (Carroll-Chindris '15)

Let (Q, I) be a bound quiver, and $A = kQ/I$ its bound quiver algebra. If A is tame, then for any irreducible component $Z \subset rep_Q(I, \mathbf{d})$ and any weight θ such that $Z_\theta^{ss} \neq \emptyset$, $\mathcal{M}(Z)_\theta^{ss}$ is a product of projective spaces.

Context and Main Theorem

- The decomposition holds for the following classes of algebras:
  - Algebras of global dimension one [Domokos-Lenzing '02],
  - Tilted Algebras [Chindris '13],
  - Quasi-tilted algebras [Bobinski '14],
  - Acyclic gentle algebras [Carroll-Chindris '15],
  - Special biserial algebras [Carroll-Chindris-Kinser-Weyman '20].

Theorem (Gilbert '23)

Let $\Lambda = kQ/I$ be a clannish algebra. Then any irreducible component of a moduli space $\mathcal{M}(\Lambda, \mathbf{d})^{ss}_{\theta}$ is isomorphic to a product of projective spaces.

- Throughout, $A = kQ/I$.
- For a fixed dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$,

$$\mathrm{rep}_Q(I, \mathbf{d}) := \{M \in \prod_{a \in Q_1} \mathrm{Mat}_{\mathbf{d}(ha) \times \mathbf{d}(ta)}(k) \mid M(r) = 0, \text{ for all } r \in I\}$$

admits a $\mathrm{GL}(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}_i)$ action via conjugation.

Definition

For $Z \subseteq \mathrm{rep}_Q(I, \mathbf{d})$ an irreducible, closed, $\mathrm{GL}(\mathbf{d})$-invariant subvariety and $\theta \in \mathbb{Z}^{Q_0}$,

$$Z_\theta^{ss} = \{M \in Z \mid \theta(\dim M) = 0 \text{ and } \theta(\dim M') \leq 0 \text{ for } M' \leq M\}$$

$$Z_\theta^{s} = \{M \in Z \mid \theta(\dim M) = 0 \text{ and } \theta(\dim M') < 0 \text{ for } 0 < M' < M\}$$

are the θ-semistable locus and θ-stable locus, respectively.

Definition

For an irreducible, θ-semistable variety $Z \subseteq \mathrm{rep}_Q(I, \mathbf{d})$ we let

$$\mathcal{M}(Z)^{ss}_{\theta} := \mathrm{Proj}(\bigoplus_{n \geq 0} \mathrm{SI}(Z)_{n\theta})$$

denote the corresponding moduli space of Z, whose points are in bijection with the closed $\mathrm{GL}(\mathbf{d})$-orbits in Z^{ss}_{θ}.

- From [CC15b], for A tame and $Z \subset \mathrm{rep}_Q(I, \mathbf{d})$ a θ-stable irreducible component, if Z is normal, then $\mathcal{M}(Z)^{ss}_{\theta}$ is either a point or $\mathbb{P}^1$.

Moduli Decomposition Theorem

Theorem (Chindris-Kinser '18)

For $Z \subset rep_Q(I, \mathbf{d})^{ss}_{\theta}$ an irreducible component, $Z = \underline{m_1 Z_1 \dot{+} \ldots \dot{+} m_r Z_r}$ a θ-stable decomposition and $\widetilde{Z} = \overline{Z_1^{\oplus m_1} \oplus \ldots \oplus Z_r^{\oplus m_r}}$ we have

1. $\mathcal{M}(\widetilde{Z})^{ss}_{\theta} = \mathcal{M}(Z)^{ss}_{\theta}$ *whenever* $\mathcal{M}(Z)^{ss}_{\theta}$ *is irreducible.*
2. *If Z_1 is an orbit closure, then*
$$\mathcal{M}(\overline{Z_1^{\oplus m_1} \oplus \ldots \oplus Z_l^{\oplus m_l}})^{ss}_{\theta} \cong \mathcal{M}(\overline{Z_2^{\oplus m_2} \oplus \ldots \oplus Z_l^{\oplus m_l}})^{ss}_{\theta}.$$
3. *There is a finite, birational map*
$$\Psi : S^{m_1}(\mathcal{M}(Z_1)^{ss}_{\theta}) \times \ldots \times S^{m_l}(\mathcal{M}(Z_l)^{ss}_{\theta}) \to \mathcal{M}(\widetilde{Z})^{ss}_{\theta}$$
which is an isomorphism when $\mathcal{M}(\widetilde{Z})^{ss}_{\theta}$ is normal.

General Result on Tame Algebras

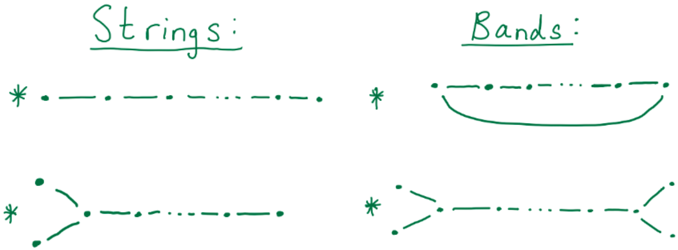
Lemma

Let $A = kQ/I$ and $B = kQ/I'$ be finite dimensional tame algebras with $I' \subset I$. Let $Z_i \subset rep_Q(I, \mathbf{d}_i)$, $1 \leq i \leq m$, be irreducible components satisfying:

- *each Z_i is Schur;*
- *none of the Z_i are orbit closures;*
- *$Hom_A(M_i, M_j) = 0$ for $i \neq j$ and general $M_i \in Z_i, M_j \in Z_j$.*

With $\mathbf{d} = \sum_{i=1}^{m} \mathbf{d}_i$, then $Z = \overline{Z_1 \oplus \ldots \oplus Z_m}$ is an irreducible component of $rep_Q(I', \mathbf{d})$ with respect to the closed embedding $rep_Q(I, \mathbf{d}) \subset rep_Q(I', \mathbf{d})$.

- Skewed-gentle algebras are gentle algebras with the additional possibility of there being idempotent loops on the vertices.
- In a similar way, clannish algebras are obtained from special biserial algebras by the ability to add these loops.
- Clannish algebras generalize special biserial algebras in that all but finitely many indecomposable representations of a clannish algebra are determined be walks of the following forms:



Proof of Main Theorem

- Let $\Lambda = kQ/I$ be a clannish algebra.

Lemma

There exists an ideal $J \subseteq I \subset kQ$ such that $\Lambda' := kQ/J$ is a skewed-gentle algebra. As such, we have an induced embedding $rep_Q(I, \mathbf{d}) \subset rep_Q(J, \mathbf{d})$.

Proposition

Let $\Lambda' = kQ/J$ be a skewed-gentle algebra and $\mathbf{d}$ be a dimension vector. If $Z \subset rep_Q(J, \mathbf{d})$ is an irreducible component, then Z is normal.

- From earlier lemma, if $Z = \overline{Z_1 \oplus \ldots \oplus Z_m} \subset \mathrm{rep}_Q(I, \mathbf{d})$ is an irreducible component where
  - each Z_i is Schur;
  - none of the Z_i are orbit closures;
  - $\mathrm{Hom}_A(M_i, M_j) = 0$ for $i \neq j$ and general $M_i \in Z_i, M_j \in Z_j$,

  we may view Z as an irreducible component of $\mathrm{rep}_Q(J, \mathbf{d})$.
- Such Z are thus normal. An argument using the moduli decomposition theorem gives the following:

Theorem

Let Λ be clannish. Then any irreducible component of a moduli space $\mathcal{M}(\Lambda, \mathbf{d})^{ss}_{\theta}$ is isomorphic to a product of projective spaces.

A Future Direction

- With kQ/I tame acyclic and

$$\mathcal{P}_A(\mathbf{d}) := \{M \in \mathrm{rep}_Q(I, \mathbf{d}) \,|\, \mathrm{pdim}_A M \leq 1\}$$

one has that $C_A(\mathbf{d}) = \overline{\mathcal{P}_A(\mathbf{d})}$ is an irreducible component. The following problem was posed by Calin Chindris

Problem

Let A be acyclic and tame. If $\mathbf{d} \in \mathbb{N}^{Q_0}$ is such that $\mathcal{P}_A(\mathbf{d}) \neq \emptyset$, describe $\mathcal{M}(C_A(\mathbf{d}))^{ss}_{\langle d, - \rangle}$.

Thank you!